Linear Algebra Test II

Surname ______________________ First Name______________________

ANSWER ALL QUESTIONS

Question 1

Let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$ be vectors in $\mathbb{R}^3$. Determine

(i) $\mathbf{u} \cdot \mathbf{v}$

$\mathbf{u} \cdot \mathbf{v} =$ ______________

[2 marks]

(ii) $\|\mathbf{u} - \mathbf{v}\|$

$\|\mathbf{u} - \mathbf{v}\| =$ ______________

[4 marks]

(iii) Let $\mathbf{w} = \begin{pmatrix} 4 \\ 4 \\ k \end{pmatrix}$. Find the value of k so that the vectors **u** and **w** are **orthogonal**.

k=_______________

[4 marks]

Question 2

(a) Let $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Show that $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ are linearly independent.

[7 marks]

(ii) Show that $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is a basis for $\mathbb{R}^3$.

[3 marks]

Question 3

Show that $\{1,\ t,\ t^2,\ t^3\}$ is a basis for P_3 which is the set of cubic polynomials.

[5 marks]

Question 4

For the given set S and matrix $\mathbf{X}$, find if $\mathbf{X}$ is in the span of S.

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad \mathbf{X} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

[6 marks]

Question 5

(i) Determine the solution of the simultaneous equations:

$$\begin{aligned} x + y + z &= 0 \\ 3x + 3y + 3z &= 0 \\ 7x + 7y + 7z &= 0 \end{aligned}$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

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[6 marks]

(ii) Determine the null space, rank, nullity of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \\ 7 & 7 & 7 \end{pmatrix}$.

null space=____________________, rank=_______ and nullity=________

[4 marks]

Question 6

Let **u**, **v** and **w** be a set of non-zero **orthogonal** vectors in $\mathbb{R}^n$. Prove that they are linearly independent.

[6 marks]

END OF TEST

Solutions to Linear Algebra Test II

2010

1.(i) $u \cdot v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = -1 - 4 - 9 = -14$ 2M

(ii) $\|u - v\| = \left\| \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} \right\|$ 2M

$= \left\| \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \right\| = \sqrt{2^2 + 4^2 + 6^2} = \sqrt{56}$

$\|u - v\| = \sqrt{56}$ 2M

(iii) The vectors are orthogonal if $\underline{u} \cdot \underline{w} = 0$. We have 1M

$\underline{u} \cdot \underline{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 4 \\ k \end{pmatrix}$

$= 4 + 8 + 3k = 0 \Rightarrow 3k = -12$

$\Rightarrow k = -4$ 3M

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2. (i) Let k_1, k_2 and k_3 be scalars such that

$k_1 \underline{u} + k_2 \underline{v} + k_3 \underline{w} = \underline{0}$ 1M

Which means we have

$k_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ 1M

Writing out the equations

$k_1 + 2k_2 + k_3 = 0$ (1)

$3k_2 + k_3 = 0$ (2)

$k_1 + k_3 = 0$ (3) 1M

From equation (3) we have $k_1 = -k_3$ 1M

2. (i) (cont'd) Substituting $k_1 = -k_3$ into equation (1) yields

$$2k_2 = 0 \Rightarrow k_2 = 0 \qquad 1M$$

Putting $k_2 = 0$ into (2) gives $k_3 = 0$. Hence substituting $k_3 = 0$ into $k_1 = -k_3$ gives $k_1 = 0$.
Since $k_1 = k_2 = k_3 = 0$ therefore $\underline{u}$, $\underline{v}$ & $\underline{w}$ are linearly independent. 2M

(ii) Since $\dim(\mathbb{R}^3) = 3$ and we have 3 linearly independent vectors therefore $\{u, v, w\}$ is a basis for $\mathbb{R}^3$.
3M

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3. The dimension of P_3 is 4 therefore we only need to show that $\{1, t, t^2, t^3\}$ ~~are~~ is linearly independent. (1M)
Let k_1, k_2, k_3 and k_4 be scalars such that

$$k_1(1) + k_2(t) + k_3(t^2) + k_4(t^3) = 0$$
1M

Equating coeffts of t^3: $k_4 = 0$

" " t^2: $k_3 = 0$

" " t: $k_2 = 0$

" " const: $k_1 = 0$ (2M)

Since $k_1 = k_2 = k_3 = k_4 = 0$ therefore $\{1, t, t^2, t^3\}$ is linearly independent which means it forms a basis for P_3. (1M)

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4. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Let k_1, k_2, k_3 and k_4 be scalars such that

$$k_1 A + k_2 B + k_3 C + k_4 D = X$$
(2M)

We have

$$k_1 A + k_2 B + k_3 C + k_4 D = k_1\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} + k_2\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + k_3\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} + k_4\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}k_1 & 0\\ k_1 & k_1\end{pmatrix} + \begin{pmatrix}0 & k_2\\ 0 & 0\end{pmatrix} + \begin{pmatrix}k_3 & k_3\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & k_4\end{pmatrix}$$

$$= \begin{pmatrix}k_1 + k_3 & k_2 + k_3\\ k_1 & k_1 + k_4\end{pmatrix} = \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$$

2M

We have

$$\left.\begin{aligned} k_1 + k_3 &= 2\\ k_2 + k_3 &= 0\\ k_1 &= 0\\ k_1 + k_4 &= 2 \end{aligned}\right\} \Rightarrow k_1 = 0,\ k_4 = 2,\ k_3 = 2 \ \& \ k_2 = -2$$

Hence X is in the span of S. 2M

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5. (i) Let $A = \begin{pmatrix}1 & 1 & 1\\ 3 & 3 & 3\\ 7 & 7 & 7\end{pmatrix}$, $\underline{x} = \begin{pmatrix}x\\ y\\ z\end{pmatrix}$ and $\underline{0} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$ 1M

Putting the matrix into reduced row echelon form gives

$$\begin{matrix}R_1\\ R_2\\ R_3\end{matrix}\begin{pmatrix}1 & 1 & 1\\ 3 & 3 & 3\\ 7 & 7 & 7\end{pmatrix} \Rightarrow \begin{matrix}R_1\\ R_2 - 3R_1\\ R_3 - 7R_1\end{matrix}\begin{pmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = R$$

2M

Solving $R\underline{x} = \underline{0}$:

$$\begin{pmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

1M

5(i) (ctnd). From the top row we have

$$x+y+z=0 \quad \text{or} \quad x=-y-z$$

Let $y=s$ and $z=t$ where $s,t \in \mathbb{R}$ then

$$\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s-t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

2M

(ii) The solution space is given by

$$\underline{x} = s\underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{=\underline{u}} + t\underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{=\underline{v}} = s\,\underline{u} + t\,\underline{v}$$ 1M

The null space of A is $N = \{s\underline{u} + t\underline{v} \mid s,t \in \mathbb{R}\}$ 1M

The vectors $\underline{u}$ & $\underline{v}$ are a basis for N so

$$\text{nullity}(A) = 2$$ 1M

The reduced row echelon form of matrix A has only one non-zero row therefore $\text{rank}(A) = 1$. 1M

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6. Proof: Let k_1, k_2 and k_3 be scalars such that

$$k_1\underline{u} + k_2\underline{v} + k_3\underline{w} = \underline{0} \quad (*)$$ 1M

Consider the inner product $\langle \underline{u}, \underline{0}\rangle = 0$. We have

$$\langle \underline{u}, \underline{0}\rangle = \langle \underline{u}, k_1\underline{u} + k_2\underline{v} + k_3\underline{w}\rangle \quad [\text{By } (*)]$$

$$= k_1\langle \underline{u},\underline{u}\rangle + k_2\underbrace{\langle \underline{u},\underline{v}\rangle}_{=0} + k_3\underbrace{\langle \underline{u},\underline{w}\rangle}_{=0}$$

$$= k_1\|\underline{u}\|^2 = 0$$ 3M

Since $\underline{u}$ is a non-zero vector therefore $k_1 = 0$. By repeating the above arguments we have $k_2 = k_3 = 0$. Hence $\{\underline{u}, \underline{v}, \underline{w}\}$ are linearly independent. ■ 2M

Misconceptions on Linear Algebra Test II - 2010 Paper

General Comments:

A large number of students still **cannot** construct proofs. Only three students out of hundred managed to prove the last result regarding orthogonal vectors in $\mathbb{R}^n$ are linearly independent. Hence question 6 was generally **not** attempted. I still don't understand why students find proofs problematic.

Additionally you should now be able to present your mathematical deductions in a systematic way but there were many scripts where it was impossible to work out what the student was trying to deduce. Always signpost your arguments so that the reader can easily understand your deductions.

Specific Comments:

<u>Question 1</u>

(i) It is amazing the number of students who can't multiply simple numbers.

(ii) A number of students wrote

$$\|\mathbf{u}-\mathbf{v}\| = \|\mathbf{u}\| - \|\mathbf{v}\| \quad \times$$

Students tend to use their intuition and think $\|\mathbf{u}-\mathbf{v}\| = \|\mathbf{u}\| - \|\mathbf{v}\|$. However this is FALSE.

<u>Question 2</u>

(i) You can also show that $\det\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\\ \mathbf{w}\end{pmatrix} \neq 0$ which means vector **u**, **v** and **w** are linearly independent.

(ii) Only need to state $\dim\left(\mathbb{R}^3\right)=3$ and since the vectors in part (i) are linearly independent so they form a basis. Many students did the tedious task of showing **u**, **v** and **w** span $\mathbb{R}^3$. To show these vectors span $\mathbb{R}^3$ is too much work for three marks.

<u>Question 4</u>

A large number of students have still not understood the meaning of the term 'span'. Span does **not** mean linear independence.

<u>Question 5</u>

A majority of students did not write the augmented matrix in row echelon form. They got has far as the following:

$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\left(\begin{array}{ccc|c}1&1&1&0\\3&3&3&0\\0&0&0&0\end{array}\right)$$

Carrying out another simple row operation $R_2 - 3R_1$ puts the augmented matrix into reduced row echelon form:

$$\begin{matrix} R_1 \\ R_2 - 3R_1 \\ R_3 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Some students who got to this still could not derive the solutions. Remember expanding the top row gives

$$x = -y - z$$

Let $y = s,\ z = t$ where s and t are any real numbers. Then $x = -s - t$.