**MATH 223, Linear Algebra**
**Fall, 2007**
**Assignment 2 Solutions**

1. Find the rank of each of the matrices below:

(a)

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

(over the rationals)

(b)

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(over $\mathcal{Z}_2$)

(c)

$$\begin{bmatrix} 1 & 2-i & 3+2i & 4-3i \\ 2+i & 1 & 4+3i & 3-2i \end{bmatrix}$$

(over $\mathcal{C}$)

**Solution:** Recall that the rank of a matrix $A$ is defined to be the number of leading ones in the reduced row echelon form of $A$ (this is the same thing as the row canonical form of $A$). Thus, to find the ranks of these matrices, we need to put them into reduced row echelon form. Using row reduction, we find that for each matrix above, the reduced row echelon form is:

(a)

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 0 & 2 & -i \\ 0 & 1 & i & 2 \end{bmatrix}$$

Notice that in each case the number of leading ones (and hence the rank) is equal to 2.

2. Consider the following $3 \times 3$ real matrices:

$$A = \begin{bmatrix} -2 & 1 & 8 \\ -1 & -1 & 7 \\ 3 & 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 0 & -7 \\ 6 & 3 & -9 \\ -2 & -2 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 6 & 3 & -1 \\ 2 & 4 & 5 \\ -1 & -1 & 8 \end{bmatrix}.$$

Find (by hand!) the following expressions:

(a) $AB$

(b) $BA$

(c) $AB - BA$

(d) $ABC$

**Solution:** Using the definition of matrix multiplication, we easily find:

(a)

$$AB = \begin{bmatrix} -20 & -13 & 5 \\ -25 & -17 & 16 \\ 7 & -8 & -21 \end{bmatrix}$$

(b)

$$BA = \begin{bmatrix} -31 & 5 & 12 \\ -42 & 3 & 33 \\ 6 & 0 & -30 \end{bmatrix}.$$

(c) Subtracting the second answer from the first gives

$$AB - BA = \begin{bmatrix} 11 & -18 & -7 \\ 17 & -20 & -17 \\ 1 & -8 & 9 \end{bmatrix}$$

(d) Finally, multiplying the first answer on the right by $C$, we find

$$ABC = \begin{bmatrix} -151 & -117 & -5 \\ -200 & -159 & 68 \\ 47 & 10 & -215 \end{bmatrix}$$

3. Show that

$$x = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is a solution to the equation

$$x^3 + 1 = 0.$$

Can you think of another $3 \times 3$ real matrix that is a solution to this equation? Bonus: find all *diagonal* complex matrices that are solutions to $x^3 + 1 = 0$.

**Solution:** Note that the 1 and 0 in $x^3+1=0$ stand for the $3\times 3$ identity matrix and the $3\times 3$ zero matrix (i.e. the $3\times 3$ matrix all of whose entries are zero) respectively. We calculate:

$$x^2 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix},$$

so we have

$$x^3 = x\cdot x^2 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

and therefore

$$x^3+1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0,$$

as required.

Observe that

$$A = -I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

satisfies $A^3 = (-1)^3\cdot I_3\cdot I_3\cdot I_3 = (-1)^3 I_3 = -I_3$, and hence that $A^3+1 = A^3+I_3 = 0$, so $A$ also satisfies the equation $x^3+1=0$.

**Bonus:** If

$$x = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

is any diagonal matrix, then

$$x^3 = \begin{bmatrix} a^3 & 0 & 0 \\ 0 & b^3 & 0 \\ 0 & 0 & c^3 \end{bmatrix}.$$

Thus, if $x^3+1=0$, then we must have $a^3=b^3=c^3=-1$. There are 3 distinct complex numbers with cube equal to -1, given by $-1, -\omega, -\omega^2$, where

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

Each of $a, b, c$ can be any one of these 3 numbers, giving a total of 27 diagonal complex matrices whose cube is $-1 = -I_3$.

4. Find elementary matrices $E_1, E_2, E_3$ such that

$$E_1 E_2 E_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

**Solution:** Note that there are many possible solutions to this problem, depending on what row operations you decide to use; we present one possible solution. Call the matrix on the right hand side above $A$. First, we determine elementary row operations that transform $A$ in to reduced row echelon form. Applying $e_1 = (R_1 - 2R_2 \to R_1)$ to $A$ gives the matrix

$$e_1(A) = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Applying $e_2 = (R_1 + 5R_3 \to R_1)$ to $e_1(A)$ yields

$$e_2(e_1(A)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally, we apply $e_3 = (R_2 - 4R_3 \to R_2)$ to $e_2(e_1(A))$ to get

$$e_3(e_2(e_1(A))) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, if we set

$$E_1 = e_1(I_3) = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = e_2(I_3) = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$E_3 = e_3(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix},$$

we have

$$E_3 E_2 E_1 A = I_3.$$

Multiplying both sides of this equation by $E_1^{-1} E_2^{-1} E_3^{-1}$ (on the left) gives

$$A = E_1^{-1} E_2^{-1} E_3^{-1}.$$

Since for $i = 1, 2, 3$ we have $E_i^{-1} = e_i^{-1}(I_3)$, we easily compute that

$$E_1^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix},$$

so

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

We reiterate that this is one of many possible solutions.

5. Show that the following sets with the indicated linear structure are vector spaces over the given field:

   (a) (Over the reals) The set $S$ of real valued functions of one real variable. Addition of two functions $f$ and $g$ is defined by

   $$(f+g)(x) = f(x) + g(x),$$

   and scalar multiplication is defined by

   $$(k \cdot f)(x) = k(f(x)).$$

   (b) (Over the complex numbers) The set $S$ of polynomials in one variable with complex coefficients, equipped with the usual addition and scalar multiplication of polynomials.

   **Solution:** To show that each set is a vector space, we must first specify what the zero vector is and how to determine additive inverses. Then we must check that the 8 axioms of being a vector space hold. In each case, we explain only how to prove that addition is commutative (i.e. $v+w = w+v$ for all vectors $v, w$). The proofs that the other axioms hold are nearly identical in nature.

   (a) The zero vector is the function $0_S$ which is *defined* by the rule $0_S(x) = 0$ for all real numbers $x$. Similarly, the additive inverse of a function $f \in S$ is the function $-f$ *defined* by the rule $-f(x) = -(f(x))$ (i.e. the value of the function $-f$ at any real number $x$ is the additive inverse of the real number $f(x)$). Let $f, g$ be any two functions. We wish to show that

   $$f + g = g + f.$$

   Since two functions are equal if and only if all of their values agree, it suffices to show that

   $$(f+g)(x) = (g+f)(x)$$

   for all real numbers $x$. By *definition* of addition of functions, the truth of this identity is equivalent to the truth of

   $$f(x) + g(x) = g(x) + f(x)$$

   for all real numbers $x$. Since $\mathcal{R}$ is a field, and therefore addition is commutative, this last identity is indeed true for all real numbers $x$.

(b) In this case, the zero vector is just the constant polynomial 0. For any polynomial $p = a_0 + a_1 t + \cdots + a_n t^n$, we define $-p$ to be the polynomial obtained by negating all the coefficients:

$$-p = (-a_0) + (-a_1)t + \cdots + (-a_n)t^n.$$

Let us check that for any two polynomials $p, q$ we have $p + q = q + p$. We can write $p = a_0 + a_1 t + \cdots + a_n t^n$ and $q = b_0 + b_1 t + \cdots + b_n t^n$ for some (possibly zero) complex numbers $a_i$ and $b_i$. Then

$$\begin{aligned} p + q &= (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n \\ &= (b_0 + a_0) + (b_1 + a_1)t + \cdots + (b_n + a_n)t^n \\ &= q + p, \end{aligned}$$

where the first and last equalities are due to the definition of addition of polynomials, and the middle equality is a consequence of the fact that addition in $\mathcal{C}$ is commutative (as $\mathcal{C}$ is a field).

6. Show that the set

$$\mathcal{Z}_6 := \{a, b, c, d, e, f\}$$

equipped with addition given by the table

| $+$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $a$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
| $b$ | $b$ | $c$ | $d$ | $e$ | $f$ | $a$ |
| $c$ | $c$ | $d$ | $e$ | $f$ | $a$ | $b$ |
| $d$ | $d$ | $e$ | $f$ | $a$ | $b$ | $c$ |
| $e$ | $e$ | $f$ | $a$ | $b$ | $c$ | $d$ |
| $f$ | $f$ | $a$ | $b$ | $c$ | $d$ | $e$ |

can not be made in to a vector space over $\mathcal{Z}_2$.

**Solution:** For this problem, we must use the following general fact: For any vector space $V$ over a field $K$, and any $v \in V$ we have

$$0 \cdot v = 0.$$

To prove this fact, set $x = 0 \cdot v$. Then

$$x = 0 \cdot v = (0 + 0)v = 0 \cdot v + 0 \cdot v = x + x.$$

Adding $-x \in V$ to both sides of this equation and using the associativity of addition and the definition of 0 yields

$$0 = x + -x = (x + x) + -x = x + (x + -x) = x + 0 = x,$$

as claimed.

Now set $V = \mathcal{Z}_6$. It follows directly from the above addition table that $V$ is equipped with a law of addition that is commutative and associative. Moreover, it is clear that $a \in V$ is the zero vector (as $a + v = v$ for all $v \in V$) and therefore, since $a$ is in every row (equivalently column, since the table is symmetric about the diagonal), we see that every $v \in V$ has an additive inverse $-v \in V$. Now if $V$ is a vector space, then we must have $1 \cdot v = v$ for all $v \in V$ (as this is part of the definition of being a vector space). Thus,

$$0 \cdot v = (1 + 1)v = 1 \cdot v + 1 \cdot v = v + v$$

for all $v \in V$, where we have used the fact that $1 + 1 = 0$ in $\mathcal{Z}_2$ for the first equality. We conclude from the above general fact that we must have $a = v + v$ for all $v \in V$. But $b + b = c \neq a$, so this is a contradiction and $V$ can not be a vector space over $\mathcal{Z}_2$.

7. Explain why each of the following subsets is or is not a subspace of the given vector space:

   (a) The subset of the complex vector space $\mathcal{C}$ consisting of those complex numbers whose absolute value is at most 1.

   (b) The subset of the real vector space of real valued functions of one variable consisting of continuous functions.

   (c) The subset of the complex vector space of polynomials with complex coefficients consisting of those polynomials all of whose roots in $\mathcal{C}$ are distinct.

   (d) The subset of the real vector space of polynomials with real coefficients consisting of those polynomials of degree at most 10.

   **Solution:** Recall that a subset $W$ of a vector space $V$ is a subspace if and only if it contains the zero vector and is closed under addition and scalar multiplication.

   (a) This subset is not closed under addition or scalar multiplication, as, for example, $3 \cdot 1 = 3$ and while $|1| = 1$ is a complex number with absolute value at most 1, the complex number 3 has absolute value $|3| = 3$, which is strictly greater than 1. Therefore, this subset is *not* a subspace.

   (b) This subset *is* a subspace because the zero function is continuous (so the subset contains the zero vector), the sum of two continuous functions is again a continuous function (so the subset is closed under addition) and the product of a real number and a continuous function is again a continuous function (so the subset is closed under scalar multiplication).

   (c) This subset is *not* a subspace because it is not closed under addition. For example, the polynomials $x^2 - 1$ and $x^2 + 1$ each have two distinct complex roots, but their sum $2x^2$ has a double root at the origin.

(d) Clearly this subset contains the zero polynomial. Moreover, it is closed under scalar multiplication because for any polynomial $p$ and any scalar $k$, the polynomial $kp$ is either zero or has the same degree as $p$. Finally, it is closed under addition since the degree of a sum of polynomials is at most the maximum of their degrees.