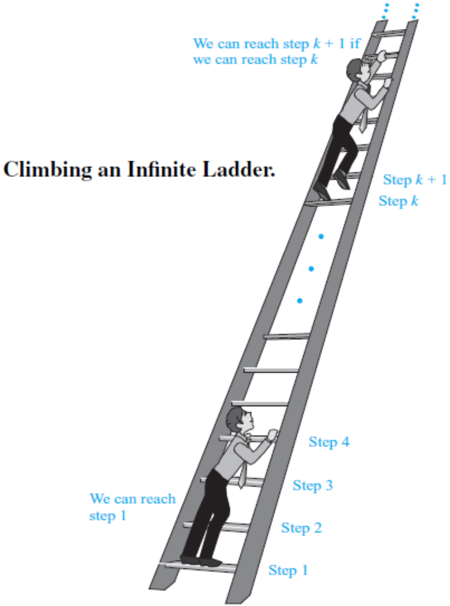
We can reach step $k + 1$ if
we can reach step k
Climbing an Infinite Ladder.
Step $k + 1$
Step k
Step 4
Step 3
We can reach
step 1
Step 2
Step 1

Mathematical Induction

PRINCIPLE OF MATHEMATICAL INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

This proof technique can be stated as

$$(P(1) \wedge \forall k (P(k) \to P(k+1))) \to \forall n P(n),$$

when the domain is the set of positive integers.

Sums of Geometric Progressions Use mathematical induction to prove formula for the sum of a finite number of terms of a geometric progression with initial term a and common ratio r:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1} \qquad \text{when } r \neq 1,$$

where n is a nonnegative integer.

Solution:

let $P(n)$ be the statement that the sum of the first $n+1$ terms of a geometric progression in this formula is correct.

BASIS STEP: $P(0)$ is true, because

$$\frac{ar^{0+1} - a}{r - 1} = \frac{ar - a}{r - 1} = \frac{a(r - 1)}{r - 1} = a.$$

Cont . . .

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, where k is an arbitrary nonnegative integer. That is, $P(k)$ is the statement that

$$a + ar + ar^2 + \cdots + ar^k = \frac{ar^{k+1} - a}{r - 1}.$$

We show that if $P(k)$ is true, then $P(k+1)$ is also true.
first add ar^{k+1} to both sides of the equality asserted by $P(k)$.

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} \overset{\text{IH}}{=} \frac{ar^{k+1} - a}{r - 1} + ar^{k+1}.$$

Rewriting the right-hand side

$$\frac{ar^{k+1} - a}{r - 1} + ar^{k+1} = \frac{ar^{k+1} - a}{r - 1} + \frac{ar^{k+2} - ar^{k+1}}{r - 1} = \frac{ar^{k+2} - a}{r - 1}.$$

Combining these last two equations gives

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+2} - a}{r - 1}.$$

This shows that if the inductive hypothesis $P(k)$ is true, then $P(k+1)$ must also be true. This completes the inductive argument.

Example 2 (Mathematical Induction)

Use mathematical induction to prove the inequality $n < 2^n$ for all positive integers n.

Solution:

Let $P(n)$ be the proposition that $n < 2^n$.

BASIS STEP: $P(1)$ is true, because $1 < 2^1 = 2$. This completes the basis step.

INDUCTIVE STEP: The inductive hypothesis $P(k)$ is the statement that $k < 2^k$.

We need to show that if $P(k)$ is true, then $P(k+1)$, that $k+1 < 2^{k+1}$, is true.

That is, we need to show that if $k < 2^k$, then $k+1 < 2^{k+1}$.

We first add 1 to both sides of $k < 2^k$

$$k + 1 \overset{\text{IH}}{<} 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}. \qquad \text{note that } 1 \leq 2^k$$

This shows that $P(k+1)$ is true, namely, that $k+1 < 2^{k+1}$ is true.

We have completed both the basis step and the inductive step, by the principle of mathematical induction we have shown that $n < 2^n$ is true for all positive integers n.

Example 3 (Mathematical Induction)

Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

Solution: let $P(n)$ denote the proposition: "$n^3 - n$ is divisible by 3."

BASIS STEP: The statement $P(1)$ is true because $1^3 - 1 = 0$ is divisible by 3. This completes the basis step.

INDUCTIVE STEP: assume that $P(k)$ is true; that is, $k^3 - k$ is divisible by 3
we must show that $(k+1)^3 - (k+1)$ is divisible by 3, is also true.

$$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1) = (k^3 - k) + 3(k^2 + k).$$

Using the inductive hypothesis, we conclude that the first term $k^3 - k$ is divisible by 3. The second term is divisible by 3 because it is 3 times an integer.

Strong Induction

STRONG INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that the proposition $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for all positive integers k.

Strong induction is sometimes called the **second principle of mathematical induction**
Show that if n is an integer greater than 1, then n can be written as the product of primes.

Solution: Let $P(n)$ be the proposition that n can be written as the product of primes.

BASIS STEP: $P(2)$ is true, because 2 can be written as the product of one prime, itself. (Note that $P(2)$ is the first case we need to establish.)

INDUCTIVE STEP: The inductive hypothesis is the assumption that $P(j)$ is true for all integers j with $2 \leq j \leq k$, that is, the assumption that j can be written as the product of primes whenever j is a positive integer at least 2 and not exceeding k. To complete the inductive step, it must be shown that $P(k+1)$ is true under this assumption, that is, that $k+1$ is the product of primes.

There are two cases to consider, namely, when $k+1$ is prime and when $k+1$ is composite. If $k+1$ is prime, we immediately see that $P(k+1)$ is true. Otherwise, $k+1$ is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b < k+1$. Because both a and b are integers at least 2 and not exceeding k, we can use the inductive hypothesis to write both a and b as the product of primes. Thus, if $k+1$ is composite, it can be written as the product of primes, namely, those primes in the factorization of a and those in the factorization of b. ◀

Recursively Defined Functions

We use two steps to define a function with the set of nonnegative integers as its domain:

BASIS STEP: Specify the value of the function at zero.

RECURSIVE STEP: Give a rule for finding its value at an integer from its values at smaller integers.

Such a definition is called a **recursive** or **inductive definition.**

Suppose that f is defined recursively by

$$f(0) = 3, \quad f(n+1) = 2f(n) + 3.$$ Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

Solution: From the recursive definition it follows that

$$f(1) = 2f(0) + 3 = 2 \cdot 3 + 3 = 9, \qquad f(2) = 2f(1) + 3 = 2 \cdot 9 + 3 = 21,$$
$$f(3) = 2f(2) + 3 = 2 \cdot 21 + 3 = 45, \quad f(4) = 2f(3) + 3 = 2 \cdot 45 + 3 = 93.$$