The Pigeonhole Principle

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Pigeonhole Principle

Suppose you have k pigeonholes and n pigeons to be placed in them. If $n > k$ ($\#\, pigeons\ >\ \#\, pigeonholes$) then at least one pigeonhole contains at least two pigeons.

In problem solving, the “pigeons” are often numbers or objects, and the “pigeonholes” are properties that the numbers/objects might possess.

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- In a group of thirteen people, must there be at least two who were born in the same month? Why?

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- In a group of thirteen people, must there be at least two who were born in the same month? Why?

A group of thirteen people, however, **must contain at least two who were born in the same month**, for there are only twelve months in a year and $13 > 12$.

Examples

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[1] 102 students

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3. How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?[1]

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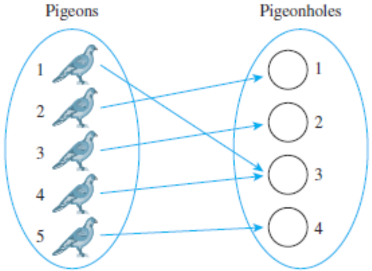
A function from one finite set to a smaller finite set cannot be one-to-one: There must be a least two elements in the domain that have the same image in the co-domain.



THE GENERALIZED PIGEONHOLE PRINCIPLE

If N objects are placed into k boxes, then there is **at least one box** containing **at least $\lceil N/k \rceil$ objects**

Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

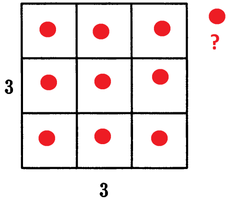
$$\text{Possible grades} = 5 = \text{ number of pigeonholes}$$

$$\text{Students required} = N = \text{ pigeonholes}$$

$$\left\lceil \frac{N}{5} \right\rceil = 6 \quad \because \text{ at least six pigeon in one hole}$$

Here $N \in \mathbf{Z}$ such that $5 \times 5 < N \leq 6 \times 5$, that is, $26 \leq N \leq 30$. The smallest such integer is $N = 26$

Show that for any set of 10 points chosen within a square whose sides are of length 3 units, there are two points in the set whose distance apart is at most $\sqrt{2}$.



10 points (our objects) chosen from the 9 unit squares. Since $10 > 9$, by (PHP), there is a unit square (box) which contains at least 2 points (objects)
The length of diagonal of unit square $\sqrt{1^2 + 1^2} = \sqrt{2}$.

REMARK # 1

If the hypothesis of (PHP) is satisfied, then the conclusion of (PHP) guarantees that there is a **(certain)** box which contains at least $\lceil \frac{N}{k} \rceil$ objects.
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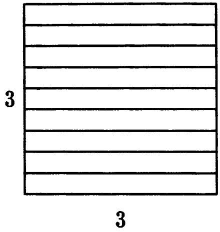
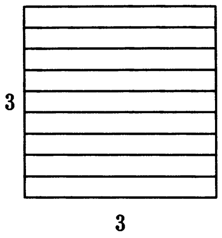
It should be noted, however, that (PHP)
does not tell us
"which box it is" and
"which $\lceil \frac{N}{k} \rceil$ objects it contains".

In above example one might divide the 3×3 square into the 9 rectangles as shown below, and apply (PHP) to reach the conclusion that there are 2 points contained in one of the 9 rectangles.



REMARK # 2 cont ⋯

In above example one might divide the 3×3 square into the 9 rectangles as shown below, and apply (PHP) to reach the conclusion that there are 2 points contained in one of the 9 rectangles.



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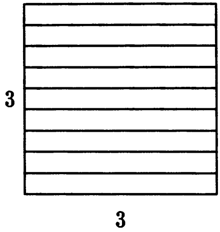

In this case, however, one would not be able to draw the conclusion that the distance between these 2 points is at most $\sqrt{2}$. So, does it mean that (PHP) is invalid? Certainly not!

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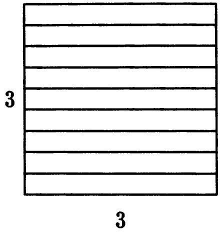

In this case, however, one would not be able to draw the conclusion that the distance between these 2 points is at most $\sqrt{2}$. So, does it mean that (PHP) is invalid? Certainly not! It simply reveals the fact that the "boxes" we create here are not appropriate!

A drawer contains ten black and ten white socks. You reach in and pull some out without looking at them. What is the least number of socks you must pull out to be sure to get a matched pair? Explain how the answer follows from the pigeonhole principle.

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