Shortest Path Algorithms

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weight function

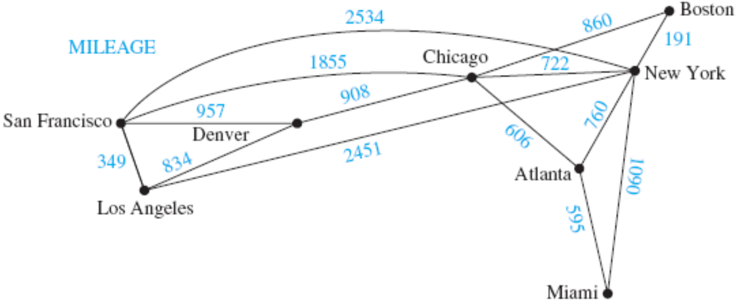
For a graph $G = (V, E)$, we define a function $w : E \to \mathbf{R}$ which assigns a real number to each edge of the graph G. This function is known as *weight function*. The real number assigned by w to an edge is called *weight of the edge*.

A graph $G = (V, E)$ in which each edge has a weight is called a *weighted graph*.

A weight may represent distance, cost or current depending upon the nature of phenomenon under study. Several types of problems involving weighted graphs arise frequently in different fields like engineering, economics and optimization. Determining a shortest path between two vertices in a graph is one such problem.

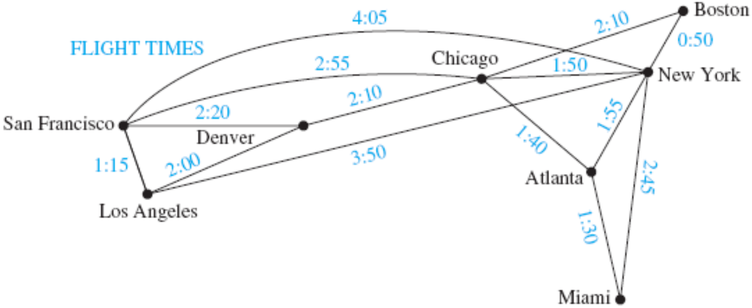
Weighted Graphs

Problems involving distances can be modeled by assigning distances between cities to the edges.



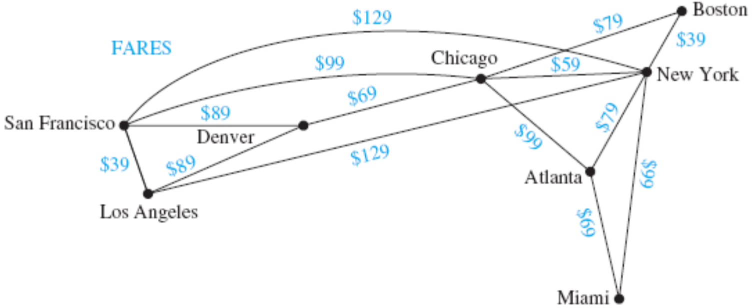
Weighted Graphs

Problems involving flight time can be modeled by assigning flight times to edges.



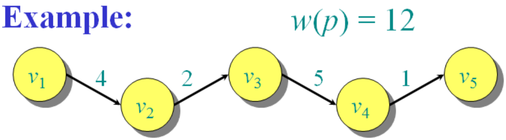
Weighted Graphs

Problems involving fares can be modeled by assigning fares to the edges.



In a weighted graph $G = (V, E)$, *weight of a walk* is the sum of the weights of the edges of the walk.
Similarly we can define the weights of paths and cycles.
Let $S = v_1, v_2, \ldots, v_k$ be a path in a weighted graph G then weight of S is given by

$$w(S) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}).$$



The term shortest path is used in general sense and it represents a path having minimum weight.

Let $G = (V, E)$ be a weighted graph. A *shortest path* from a vertex v to a vertex v' is defined as any $v - v'$ path S with weight $w(S) = d(v, v')$ where $d(v, v')$ is given by

$$d(v, v') = \begin{cases} \min\{w(\widetilde{P}) \mid \widetilde{P} \text{ is a } v - v' \text{ path}\} & \text{if } v' \text{ is reachable from } v \\ +\infty & \text{otherewise.} \end{cases}$$

Shortest Path Problems

- How can we find the shortest route between two points on a road map?

Model the problem as a graph problem

- Road map is a weighted graph

$$\begin{aligned} \textit{vertices} &= \text{cities} \\ \textit{edges} &= \text{road segments between cities} \\ \textit{edge weights} &= \text{road distances} \end{aligned}$$

Goal: find a shortest path between two vertices (cities)

Variants of Shortest Path

Single-source shortest paths

Find a shortest path from a given source vertex s to each vertex $v \in V$

Single-destination shortest paths

Find a shortest path to a given destination vertex t from each vertex v

Single-pair shortest path

Find a shortest path from u to v for given vertices u and v

All-pairs shortest-paths

Find a shortest path from u to v for every pair of vertices u and v

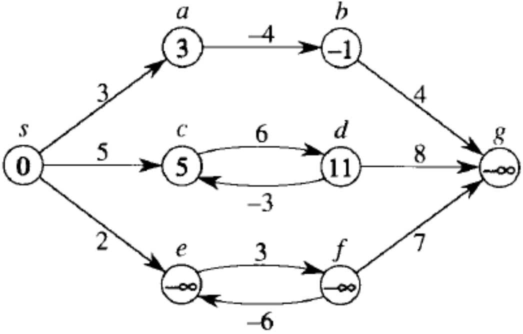
Negative Cycle

It is obvious from the definition of the weight function that weight of an edge is a real number which may be positive are negative.

Negative weights are not merely a mathematical curiosity; they arise in a natural way when we reduce other problems to shortest path problems.

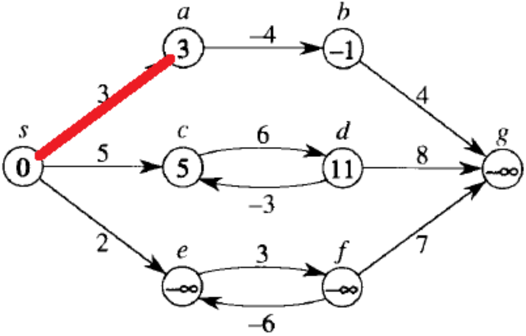
A cycle C in a weighted graph with $w(C) < 0$ is called a *negative cycle*.

Negative Cycle



Negative Cycle

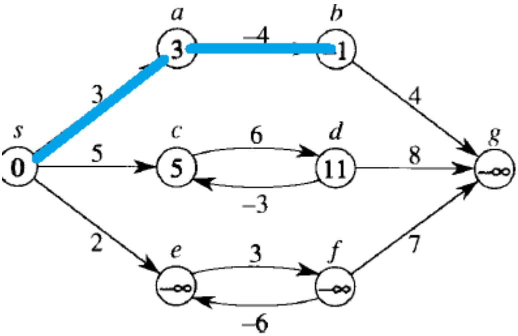
Only one path from s to a



$$W(S_{sa}) = w(s, a) = 3$$

Negative Cycle

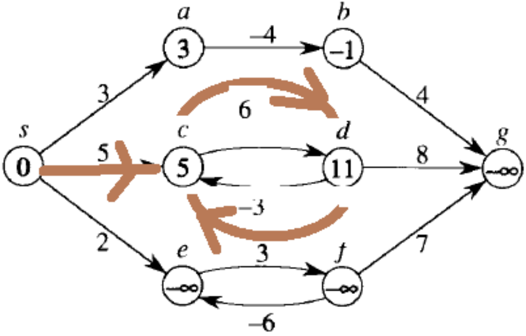
Only one path from s to b i.e. s, a, b



$$W(S_{sb}) = w(s,a) + w(a,b) = 3 + (-4) = -1$$

Negative Cycle

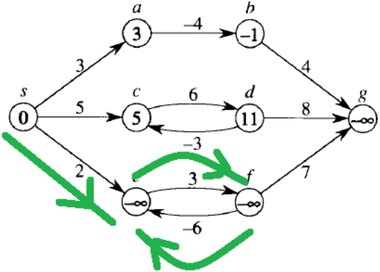
Infinity many paths from s to c i.e. s, c; s, c, d, c; s, c, d, c, d, c and so on.



Because the cycle c, d, c has weight 3>0, so the shortest path from s to c has weight 5.
The shortest path from s to d has weight 11.

Negative Cycle

Infinity many paths from s to e i.e. s, e; s, e, f, e; $s, , e, f, e, f, e$ and so on.



As the weight of cycle e, f, e is -3 (negative). Therefore, there is no shortest path from s to e.
Because, otherwise, by traversing the negative weight cycle gives us a lesser weight path each time.

Assignment Quiz 2

Please write your name, row number and assignment quiz 2 on your answer sheet.

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1. Find a recurrence relation to describe the amount left to pay on a loan of £10000, with interest charged at 1.5% per month and fixed monthly payment of £250.
2. Find a recurrence relation to describe the amount of water in a swimming pool of volume 7 50,000 liters if 0.05% per day is lost to evaporation but 350 liters is added daily.
3. There are five points inside an equilateral triangle of side length 2. Show that at least two of the points are within 1 unit distance from each other.
4. How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16?

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   (b) have exactly three digits that are 9s?
3. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$. Show that R is an equivalence relation.
4. R is the relation defined on $\mathbf{Z}$ as follows: For all $(m, n) \in \mathbf{Z}$, $mRn \Leftrightarrow 4|(m^2 + n^2)$.

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