Some Definitions in Graph Theory

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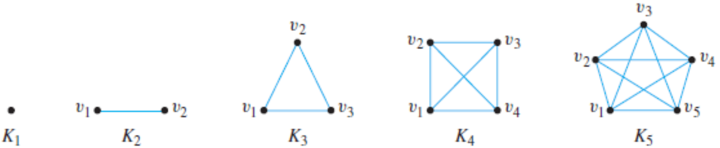

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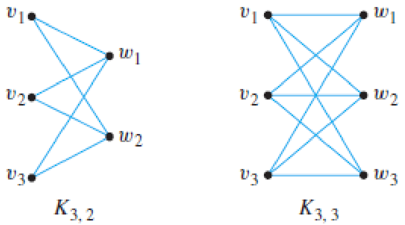
The complete bipartite graphs $K_{3,2}$ and $K_{3,3}$ are



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A graph H is said to be a subgraph of a graph G if, and only if,
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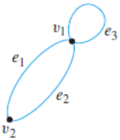
every edge in H is also an edge in G,
and every edge in H has the same endpoints as it has in G.

Subgraph



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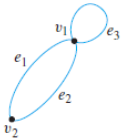

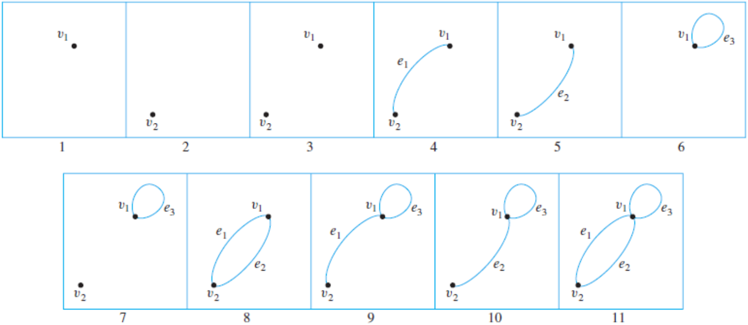

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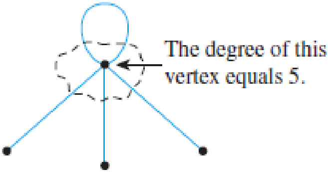

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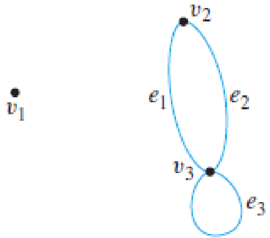
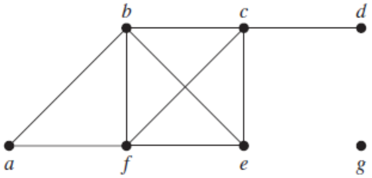


- $deg(v_1) =$
- $deg(v_2) =$
- $deg(v_3) =$
- Total degree of $G = deg(v_1) + deg(v_2) + deg(v_3) =$

Degree



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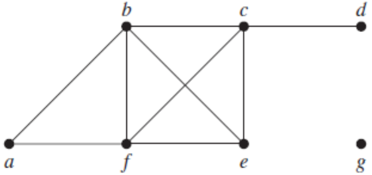

A vertex is **pendant** if and only if it has degree one.

The Handshake Theorem

Imagine a group of people shake hands with each at a party. If the numbers of handshake experienced by each person are added together, the sum will equal twice the total number of handshakes.

If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G.

Specifically, if the vertices of G are $v_1, v_2, ..., v_n$, where n is a nonnegative integer, then the total

$$\text{degree of } G = deg(v_1) + \cdots + deg(v_n) = 2(\text{ the number of edges of } G).$$

Total Degree of a Graph?

The total degree of a graph is even.

Draw a graph with the specified properties or show that no such graph exists.

1. A graph with four vertices of degrees 1, 1, 2, and 3
2. A graph with four vertices of degrees 1, 1, 3, and 3
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Is it possible in a group of nine people for each to be friends with exactly five others?

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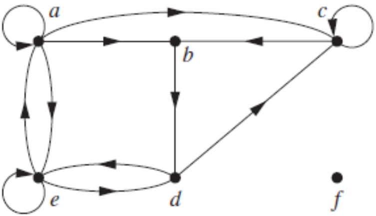
(Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

Degree, Directed Graphs

Find the in-degree and out-degree of each vertex in the graph G with directed edges shown



G

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Theorem

Let $G = (V, E)$ *be a graph with directed edges. Then*

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There are many properties of a graph with directed edges that do not depend on the direction of its edges. Consequently, it is often useful to ignore these directions. The undirected graph that results from ignoring directions of edges is called the **underlying undirected graph**. A graph with directed edges and its underlying undirected graph have the same number of edges.

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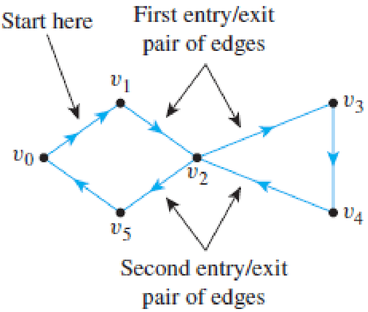

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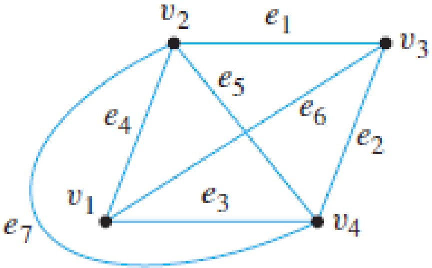

The Seven Bridges of Königsberg

The town of Königsberg in Prussia (now Kaliningrad in Russia) was built at a point where two branches of the Pregel River came together. It consisted of an island and some land along the river banks.These were connected by seven bridges as shown

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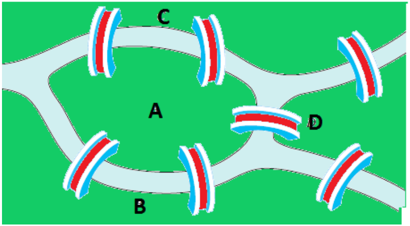

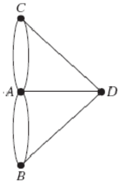

Is it possible for a person to take a walk around town, starting and ending at the same location and crossing each of the seven bridges exactly once?

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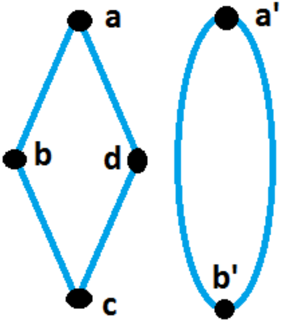

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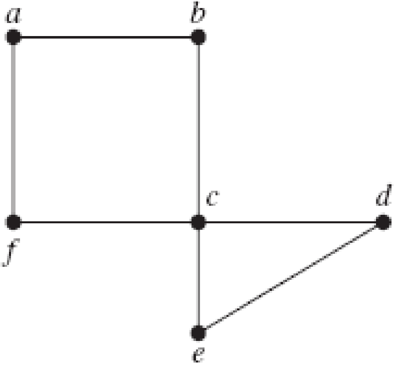
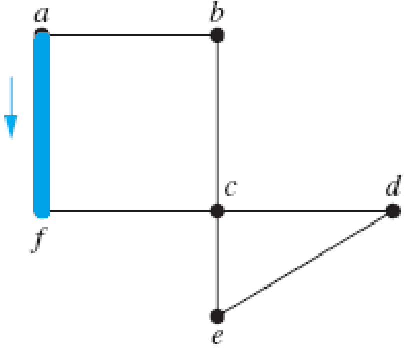
How to find Euler Circuit for a Graph??

Finding Euler Circuit



G

Finding Euler Circuit



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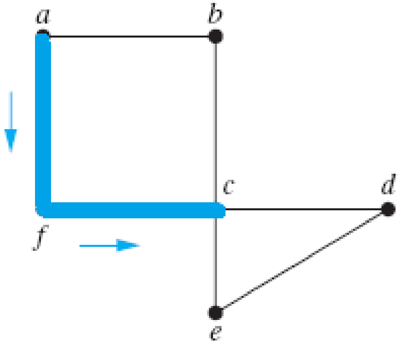

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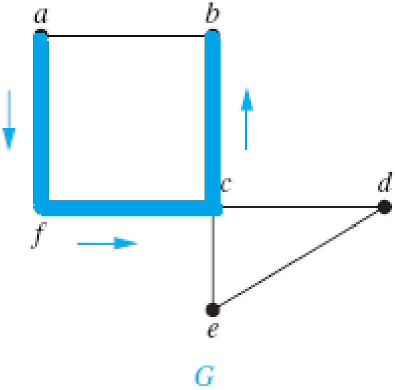

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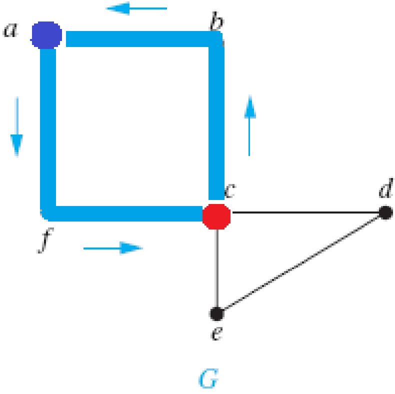
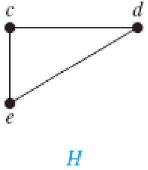
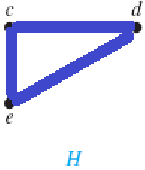


circuit a, f, c, b, a

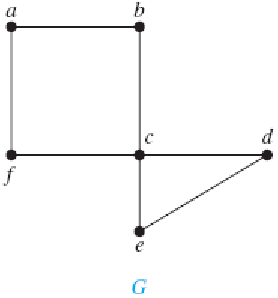
Finding Euler Circuit





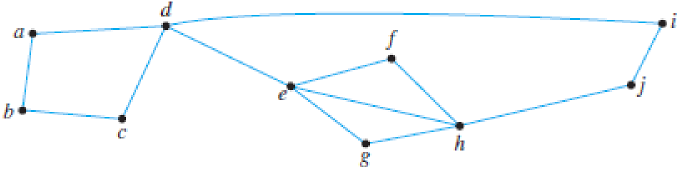
c, d, e, c

Finding Euler Circuit



we obtain the circuit a, f, c, d, e, c, b, a.

Finding an Euler Circuit



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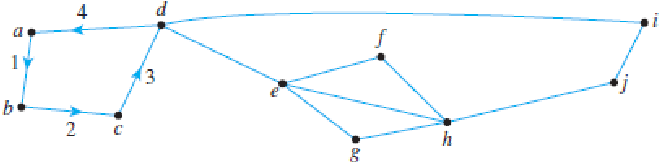

C: $abcda$.

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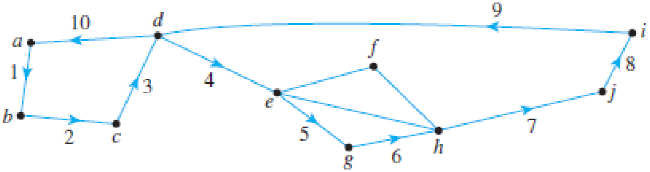

C': $deghjid$.

C'': $abcdeghjida$.

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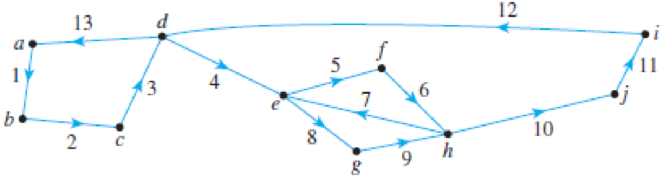


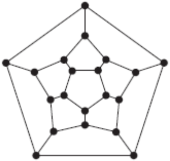
C': $efhe$.

C'': $abcdefheghjida$.

Hamiltonian circuit

Given a graph G,a **Hamiltonian** circuit for G is a simple circuit that includes every vertex of G. That is,a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, except for the first and the last, which are the same.
That is, the simple path $x_0, x_1, ..., x_{n-1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, ..., x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$, and the simple circuit $x_0, x_1, ..., x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, ..., x_{n-1}, x_n$ is a Hamilton path.

Hamiltonian circuit



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