PROBLEM SET 11.1

1–5 PERIOD, FUNDAMENTAL PERIOD

The *fundamental period* is the smallest positive period. Find it for

1. $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cos \pi x$, $\sin \pi x$, $\cos 2\pi x$, $\sin 2\pi x$

2. $\cos nx$, $\sin nx$, $\cos \dfrac{2\pi x}{k}$, $\sin \dfrac{2\pi x}{k}$, $\cos \dfrac{2\pi nx}{k}$, $\sin \dfrac{2\pi nx}{k}$

3. If $f(x)$ and $g(x)$ have period p, show that $h(x) = af(x) + bg(x)$ (a, b, constant) has the period p. Thus all functions of period p form a **vector space**.

4. Change of scale. If $f(x)$ has period p, show that $f(ax)$, $a \neq 0$, and $f(x/b)$, $b \neq 0$, are periodic functions of x of periods p/a and bp, respectively. Give examples.

5. Show that $f =$ const is periodic with any period but has no fundamental period.

6–10 GRAPHS OF 2π–PERIODIC FUNCTIONS

Sketch or graph $f(x)$ which for $-\pi < x < \pi$ is given as follows.

6. $f(x) = |x|$

7. $f(x) = |\sin x|$, $f(x) = \sin |x|$

8. $f(x) = e^{-|x|}$, $f(x) = |e^{-x}|$

9. $f(x) = \begin{cases} x & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } 0 < x < \pi \end{cases}$

10. $f(x) = \begin{cases} -\cos^2 x & \text{if } -\pi < x < 0 \\ \cos^2 x & \text{if } 0 < x < \pi \end{cases}$

11. Calculus review. Review integration techniques for integrals as they are likely to arise from the Euler formulas, for instance, definite integrals of $x \cos nx$, $x^2 \sin nx$, $e^{-2x} \cos nx$, etc.

12–21 FOURIER SERIES

Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π. Show the details of your work. Sketch or graph the partial sums up to that including $\cos 5x$ and $\sin 5x$.

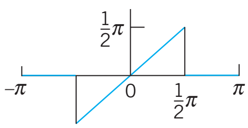
12. $f(x)$ in Prob. 6

13. $f(x)$ in Prob. 9

14. $f(x) = x^2$ $(-\pi < x < \pi)$

15. $f(x) = x^2$ $(0 < x < 2\pi)$

16.



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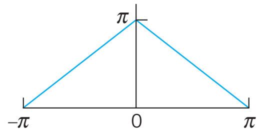

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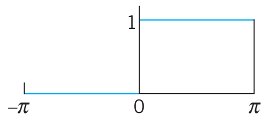

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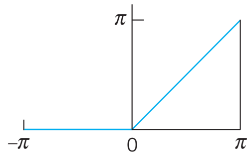

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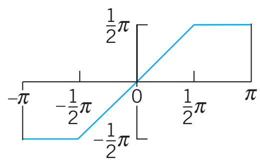

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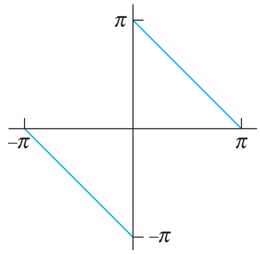


22. CAS EXPERIMENT. Graphing. Write a program for graphing partial sums of the following series. Guess from the graph what $f(x)$ the series may represent. Confirm or disprove your guess by using the Euler formulas.

(a) $2(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots)$
$\qquad - 2(\frac{1}{2} \sin 2x + \frac{1}{4} \sin 4x + \frac{1}{6} \sin 6x \cdots)$

(b) $\dfrac{1}{2} + \dfrac{4}{\pi^2}\left(\cos x + \dfrac{1}{9} \cos 3x + \dfrac{1}{25} \cos 5x + \cdots\right)$

(c) $\frac{2}{3}\pi^2 + 4(\cos x - \frac{1}{4} \cos 2x + \frac{1}{9} \cos 3x - \frac{1}{16} \cos 4x + - \cdots)$

23. Discontinuities. Verify the last statement in Theorem 2 for the discontinuities of $f(x)$ in Prob. 21.

24. CAS EXPERIMENT. Orthogonality. Integrate and graph the integral of the product $\cos mx \cos nx$ (with various integer m and n of your choice) from $-a$ to a as a function of a and conclude orthogonality of $\cos mx$

and cos nx ($m \neq n$) for $a = \pi$ from the graph. For what m and n will you get orthogonality for $a = \pi/2, \pi/3, \pi/4$? Other a? Extend the experiment to cos mx sin nx and sin mx sin nx.

25. **CAS EXPERIMENT. Order of Fourier Coefficients.** The order seems to be $1/n$ if f is discontinous, and $1/n^2$ if f is continuous but $f' = df/dx$ is discontinuous, $1/n^3$ if f and f' are continuous but f'' is discontinuous, etc. Try to verify this for examples. Try to prove it by integrating the Euler formulas by parts. What is the practical significance of this?

11.2 Arbitrary Period. Even and Odd Functions. Half-Range Expansions

We now expand our initial basic discussion of Fourier series.

Orientation. This section concerns three topics:

1. Transition from period 2π to any period $2L$, for the function f, simply by a transformation of scale on the x-axis.
2. Simplifications. Only cosine terms if f is even ("Fourier cosine series"). Only sine terms if f is odd ("Fourier sine series").
3. Expansion of f given for $0 \leqq x \leqq L$ in two Fourier series, one having only cosine terms and the other only sine terms ("half-range expansions").

1. From Period 2π to Any Period $p = 2L$

Clearly, periodic functions in applications may have any period, not just 2π as in the last section (chosen to have simple formulas). The notation $p = 2L$ for the period is practical because L will be a length of a violin string in Sec. 12.2, of a rod in heat conduction in Sec. 12.5, and so on.

The transition from period 2π to be period $p = 2L$ is effected by a suitable change of scale, as follows. Let $f(x)$ have period $p = 2L$. Then we can introduce a new variable v such that $f(x)$, as a function of v, has period 2π. If we set

$$(1) \qquad \text{(a)} \quad x = \frac{p}{2\pi} v, \qquad \text{so that} \qquad \text{(b)} \quad v = \frac{2\pi}{p} x = \frac{\pi}{L} x$$

then $v = \pm\pi$ corresponds to $x = \pm L$. This means that f, as a function of v, has period 2π and, therefore, a Fourier series of the form

$$(2) \qquad f(x) = f\left(\frac{L}{\pi} v\right) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv)$$

with coefficients obtained from (6) in the last section

$$(3) \qquad \begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) dv, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) \cos nv \, dv, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) \sin nv \, dv. \end{aligned}$$

We insert these two results into the formula for a_n. The sine terms cancel and so does a factor L^2. This gives

$$a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right).$$

Thus,

$$a_2 = -16k/(2^2\pi^2), \qquad a_6 = -16k/(6^2\pi^2), \qquad a_{10} = -16k/(10^2\pi^2), \cdots$$

and $a_n = 0$ if $n \neq 2, 6, 10, 14, \cdots$. Hence the first half-range expansion of $f(x)$ is (Fig. 272a)

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + \cdots\right).$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$.

(b) ***Odd periodic extension.*** Similarly, from (6**) we obtain

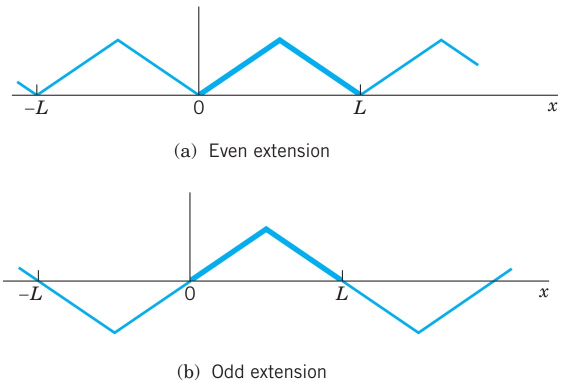
$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}. \tag{5}$$

Hence the other half-range expansion of $f(x)$ is (Fig. 272b)

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - + \cdots\right).$$

The series represents the odd periodic extension of $f(x)$, of period $2L$.

Basic applications of these results will be shown in Secs. 12.3 and 12.5. ■



(a) Even extension

(b) Odd extension

Fig. 272. Periodic extensions of $f(x)$ in Example 6

PROBLEM SET 11.2

1–7 EVEN AND ODD FUNCTIONS

Are the following functions even or odd or neither even nor odd?

1. e^x, $\quad e^{-|x|}$, $\quad x^3 \cos nx$, $\quad x^2 \tan \pi x$, $\quad \sinh x - \cosh x$

2. $\sin^2 x$, $\quad \sin(x^2)$, $\quad \ln x$, $\quad x/(x^2 + 1)$, $\quad x \cot x$

3. Sums and products of even functions

4. Sums and products of odd functions

5. Absolute values of odd functions

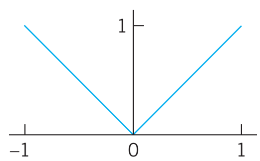
6. Product of an odd times an even function

7. Find all functions that are both even and odd.

8–17 FOURIER SERIES FOR PERIOD $p = 2L$

Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.

8.



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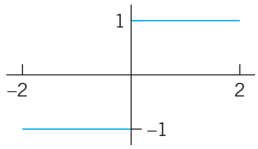

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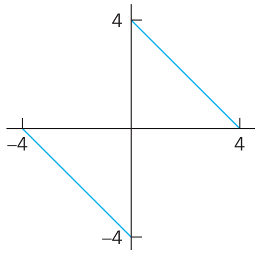
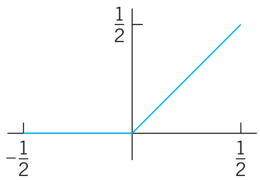


11. $f(x) = x^2 \quad (-1 < x < 1), \quad p = 2$

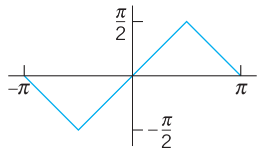
12. $f(x) = 1 - x^2/4 \quad (-2 < x < 2), \quad p = 4$

13.



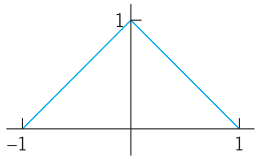
14. $f(x) = \cos \pi x \quad (-\frac{1}{2} < x < \frac{1}{2}), \quad p = 1$

15.



16. $f(x) = x|x| \quad (-1 < x < 1), \quad p = 2$

17.



18. Rectifier. Find the Fourier series of the function obtained by passing the voltage $v(t) = V_0 \cos 100\pi t$ through a half-wave rectifier that clips the negative half-waves.

19. Trigonometric Identities. Show that the familiar identities $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$ and $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$ can be interpreted as Fourier series expansions. Develop $\cos^4 x$.

20. Numeric Values. Using Prob. 11, show that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{1}{6}\pi^2$.

21. CAS PROJECT. Fourier Series of 2L-Periodic Functions. (a) Write a program for obtaining partial sums of a Fourier series (5).

(b) Apply the program to Probs. 8–11, graphing the first few partial sums of each of the four series on common axes. Choose the first five or more partial sums until they approximate the given function reasonably well. Compare and comment.

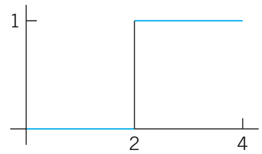
22. Obtain the Fourier series in Prob. 8 from that in Prob. 17.

23–29 HALF-RANGE EXPANSIONS

Find **(a)** the Fourier cosine series, **(b)** the Fourier sine series. Sketch $f(x)$ and its two periodic extensions. Show the details.

23.

24.



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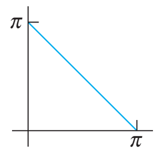

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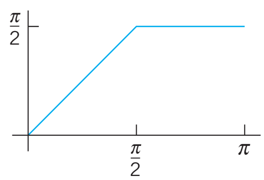

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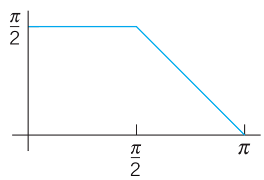

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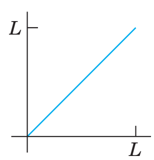

29. $f(x) = \sin x \ (0 < x < \pi)$

30. Obtain the solution to Prob. 26 from that of Prob. 27.

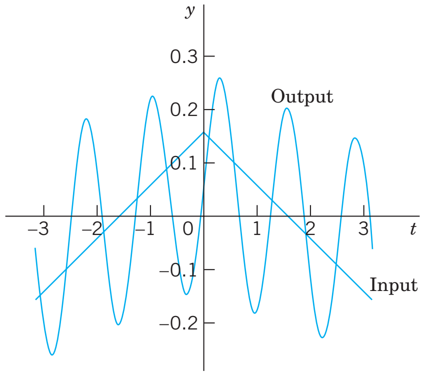

Fig. 277. Input and steady-state output in Example 1

PROBLEM SET 11.3

1. **Coefficients C_n.** Derive the formula for C_n from A_n and B_n.
2. **Change of spring and damping.** In Example 1, what happens to the amplitudes C_n if we take a stiffer spring, say, of $k = 49$? If we increase the damping?
3. **Phase shift.** Explain the role of the B_n's. What happens if we let $c \to 0$?
4. **Differentiation of input.** In Example 1, what happens if we replace $r(t)$ with its derivative, the rectangular wave? What is the ratio of the new C_n to the old ones?
5. **Sign of coefficients.** Some of the A_n in Example 1 are positive, some negative. All B_n are positive. Is this physically understandable?

6–11 GENERAL SOLUTION

Find a general solution of the ODE $y'' + \omega^2 y = r(t)$ with $r(t)$ as given. Show the details of your work.

6. $r(t) = \sin \alpha t + \sin \beta t,\ \omega^2 \neq \alpha^2, \beta^2$
7. $r(t) = \sin t,\ \omega = 0.5, 0.9, 1.1, 1.5, 10$
8. **Rectifier.** $r(t) = \pi/4\ |\cos t|$ if $-\pi < t < \pi$ and $r(t + 2\pi) = r(t),\ |\omega| \neq 0, 2, 4, \cdots$
9. What kind of solution is excluded in Prob. 8 by $|\omega| \neq 0, 2, 4, \cdots$?
10. **Rectifier.** $r(t) = \pi/4\ |\sin t|$ if $0 < t < 2\pi$ and $r(t + 2\pi) = r(t),\ |\omega| \neq 0, 2, 4, \cdots$
11. $r(t) = \begin{cases} -1 & \text{if } -\pi < t < 0 \\ 1 & \text{if } 0 < t < \pi, \end{cases} \quad |\omega| \neq 1, 3, 5, \cdots$
12. **CAS Program.** Write a program for solving the ODE just considered and for jointly graphing input and output of an initial value problem involving that ODE. Apply the program to Probs. 7 and 11 with initial values of your choice.

13–16 STEADY-STATE DAMPED OSCILLATIONS

Find the steady-state oscillations of $y'' + cy' + y = r(t)$ with $c > 0$ and $r(t)$ as given. Note that the spring constant is $k = 1$. Show the details. In Probs. 14–16 sketch $r(t)$.

13. $r(t) = \sum_{n=1}^{N} (a_n \cos nt + b_n \sin nt)$
14. $r(t) = \begin{cases} -1 & \text{if } -\pi < t < 0 \\ 1 & \text{if } 0 < t < \pi \end{cases}$ and $r(t + 2\pi) = r(t)$
15. $r(t) = t(\pi^2 - t^2)$ if $-\pi < t < \pi$ and $r(t + 2\pi) = r(t)$
16. $r(t) = \begin{cases} t & \text{if } -\pi/2 < t < \pi/2 \\ \pi - t & \text{if } \pi/2 < t < 3\pi/2 \end{cases}$ and $r(t + 2\pi) = r(t)$

17–19 *RLC*-CIRCUIT

Find the steady-state current $I(t)$ in the RLC-circuit in Fig. 275, where $R = 10\ \Omega$, $L = 1$ H, $C = 10^{-1}$ F and with $E(t)$ V as follows and periodic with period 2π. Graph or sketch the first four partial sums. Note that the coefficients of the solution decrease rapidly. *Hint.* Remember that the ODE contains $E'(t)$, not $E(t)$, cf. Sec. 2.9.

17. $E(t) = \begin{cases} -50t^2 & \text{if } -\pi < t < 0 \\ 50t^2 & \text{if } 0 < t < \pi \end{cases}$

18. $E(t) = \begin{cases} 100\,(t - t^2) & \text{if} \quad -\pi < t < 0 \\ 100\,(t + t^2) & \text{if} \quad 0 < t < \pi \end{cases}$

19. $E(t) = 200t(\pi^2 - t^2) \quad (-\pi < t < \pi)$

20. CAS EXPERIMENT. Maximum Output Term. Graph and discuss outputs of $y'' + cy' + ky = r(t)$ with $r(t)$ as in Example 1 for various c and k with emphasis on the maximum C_n and its ratio to the second largest $|C_n|$.

11.4 Approximation by Trigonometric Polynomials

Fourier series play a prominent role not only in differential equations but also in **approximation theory**, an area that is concerned with approximating functions by other functions—usually simpler functions. Here is how Fourier series come into the picture.

Let $f(x)$ be a function on the interval $-\pi \leqq x \leqq \pi$ that can be represented on this interval by a Fourier series. Then the ***N*th partial sum** of the Fourier series

$$f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \tag{1}$$

is an approximation of the given $f(x)$. In (1) we choose an arbitrary N and keep it fixed. Then we ask whether (1) is the "best" approximation of f by a **trigonometric polynomial of the same degree N**, that is, by a function of the form

$$F(x) = A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) \qquad (N \text{ fixed}). \tag{2}$$

Here, "best" means that the "error" of the approximation is as small as possible.

Of course we must first define what we mean by the **error** of such an approximation. We could choose the maximum of $|f(x) - F(x)|$. But in connection with Fourier series it is better to choose a definition of error that measures the goodness of agreement between f and F *on the whole interval* $-\pi \leqq x \leqq \pi$. This is preferable since the sum f of a Fourier series may have jumps: F in Fig. 278 is a good overall approximation of f, but the maximum of $|f(x) - F(x)|$ (more precisely, the *supremum*) is large. We choose

$$E = \int_{-\pi}^{\pi} (f - F)^2\, dx. \tag{3}$$

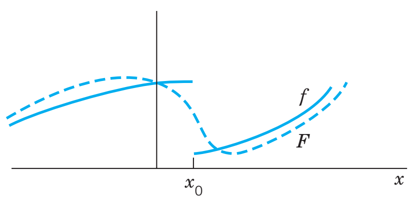

Fig. 278. Error of approximation

Example 3 confirms, from this new perspective, that the trigonometric system underlying the Fourier series is orthogonal, as we knew from Sec. 11.1.

EXAMPLE 4 Application of Theorem 1. Orthogonlity of the Legendre Polynomials

Legendre's equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ may be written

$$[(1 - x^2)y']' + \lambda y = 0 \qquad \lambda = n(n + 1).$$

Hence, this is a Sturm–Liouville equation (1) with $p = 1 - x^2$, $q = 0$, and $r = 1$. Since $p(-1) = p(1) = 0$, we need no boundary conditions, but have a "***singular*" *Sturm–Liouville problem*** on the interval $-1 \leqq x \leqq 1$. We know that for $n = 0, 1, \cdots$, hence $\lambda = 0, 1 \cdot 2, 2 \cdot 3, \cdots$, the Legendre polynomials $P_n(x)$ are solutions of the problem. Hence these are the eigenfunctions. From Theorem 1 it follows that they are orthogonal on that interval, that is,

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \qquad (m \neq n). \tag{10}$$ ■

What we have seen is that the trigonometric system, underlying the Fourier series, is a solution to a Sturm–Liouville problem, as shown in Example 1, and that this trigonometric system is orthogonal, which we knew from Sec. 11.1 and confirmed in Example 3.

PROBLEM SET 11.5

1. Proof of Theorem 1. Carry out the details in Cases 3 and 4.

2–6 ORTHOGONALITY

2. Normalization of eigenfunctions y_m of (1), (2) means that we multiply y_m by a nonzero constant c_m such that $c_m y_m$ has norm 1. Show that $z_m = cy_m$ with *any* $c \neq 0$ is an eigenfunction for the eigenvalue corresponding to y_m.

3. Change of *x*. Show that if the functions $y_0(x), y_1(x), \cdots$ form an orthogonal set on an interval $a \leqq x \leqq b$ (with $r(x) = 1$), then the functions $y_0(ct + k), y_1(ct + k), \cdots, c > 0$, form an orthogonal set on the interval $(a - k)/c \leqq t \leqq (b - k)/c$.

4. Change of *x*. Using Prob. 3, derive the orthogonality of $1, \cos \pi x, \sin \pi x, \cos 2\pi x, \sin 2\pi x, \cdots$ on $-1 \leqq x \leqq 1$ $(r(x) = 1)$ from that of $1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots$ on $-\pi \leqq x \leqq \pi$.

5. Legendre polynomials. Show that the functions $P_n(\cos \theta)$, $n = 0, 1, \cdots$, from an orthogonal set on the interval $0 \leqq \theta \leqq \pi$ with respect to the weight function $\sin \theta$.

6. Tranformation to Sturm–Liouville form. Show that $y'' + fy' + (g + \lambda h)y = 0$ takes the form (1) if you set $p = \exp(\int f\,dx)$, $q = pg$, $r = hp$. Why would you do such a transformation?

7–15 STURM–LIOUVILLE PROBLEMS

Find the eigenvalues and eigenfunctions. Verify orthogonality. Start by writing the ODE in the form (1), using Prob. 6. Show details of your work.

7. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(10) = 0$

8. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$

9. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$

10. $y'' + \lambda y = 0, \quad y(0) = y(1), \quad y'(0) = y'(1)$

11. $(y'/x)' + (\lambda + 1)y/x^3 = 0, \quad y(1) = 0, \quad y(e^{\pi}) = 0.$ (Set $x = e^t$.)

12. $y'' - 2y' + (\lambda + 1)y = 0, \quad y(0) = 0, \quad y(1) = 0$

13. $y'' + 8y' + (\lambda + 16)y = 0, \quad y(0) = 0, \quad y(\pi) = 0$

14. TEAM PROJECT. Special Functions. Orthogonal polynomials play a great role in applications. For this reason, Legendre polynomials and various other orthogonal polynomials have been studied extensively; see Refs. [GenRef1], [GenRef10] in App. 1. Consider some of the most important ones as follows.

(a) Chebyshev polynomials[6] of the first and second kind are defined by

$$T_n(x) = \cos(n \arccos x)$$

$$U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}}$$

respectively, where $n = 0, 1, \cdots$. Show that

$$T_0 = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x,$$

$$U_0 = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x.$$

Show that the Chebyshev polynomials $T_n(x)$ are orthogonal on the interval $-1 \leqq x \leqq 1$ with respect to the weight function $r(x) = 1/\sqrt{1-x^2}$. (*Hint.* To evaluate the integral, set $\arccos x = \theta$.) Verify that $T_n(x)$, $n = 0, 1, 2, 3$, satisfy the **Chebyshev equation**

$$(1-x^2)y'' - xy' + n^2 y = 0.$$

(b) Orthogonality on an infinite interval: Laguerre polynomials[7] are defined by $L_0 = 1$, and

$$L_n(x) = \frac{e^x}{n!}\frac{d^n(x^n e^{-x})}{dx^n}, \qquad n = 1, 2, \cdots.$$

Show that

$$L_n(x) = 1 - x, \qquad L_2(x) = 1 - 2x + x^2/2,$$

$$L_3(x) = 1 - 3x + 3x^2/2 - x^3/6.$$

Prove that the Laguerre polynomials are orthogonal on the positive axis $0 \leqq x < \infty$ with respect to the weight function $r(x) = e^{-x}$. *Hint.* Since the highest power in L_m is x^m, it suffices to show that $\int e^{-x}x^k L_n\,dx = 0$ for $k < n$. Do this by k integrations by parts.

11.6 Orthogonal Series. Generalized Fourier Series

Fourier series are made up of the trigonometric system (Sec. 11.1), which is orthogonal, and orthogonality was essential in obtaining the Euler formulas for the Fourier coefficients. Orthogonality will also give us coefficient formulas for the desired generalized Fourier series, including the Fourier–Legendre series and the Fourier–Bessel series. This generalization is as follows.

Let $y_0, y_1, y_2, \cdots$ be orthogonal with respect to a weight function $r(x)$ on an interval $a \leqq x \leqq b$, and let $f(x)$ be a function that can be represented by a convergent series

(1) $$f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + \cdots.$$

This is called an **orthogonal series**, **orthogonal expansion**, or **generalized Fourier series**. If the y_m are the eigenfunctions of a Sturm–Liouville problem, we call (1) an **eigenfunction expansion**. In (1) we use again m for summation since n will be used as a fixed order of Bessel functions.

Given $f(x)$, we have to determine the coefficients in (1), called the **Fourier constants** *of $f(x)$ with respect to $y_0, y_1, \cdots$*. Because of the orthogonality, this is simple. Similarly to Sec. 11.1, we multiply both sides of (1) by $r(x)y_n(x)$ (n ***fixed***) and then integrate on

[6]PAFNUTI CHEBYSHEV (1821–1894), Russian mathematician, is known for his work in approximation theory and the theory of numbers. Another transliteration of the name is TCHEBICHEF.

[7]EDMOND LAGUERRE (1834–1886), French mathematician, who did research work in geometry and in the theory of infinite series.

PROBLEM SET 11.7

1–6 EVALUATION OF INTEGRALS

Show that the integral represents the indicated function. *Hint*. Use (5), (10), or (11); the integral tells you which one, and its value tells you what function to consider. Show your work in detail.

1. $$\int_0^\infty \frac{\cos xw + w\sin xw}{1+w^2}\,dx = \begin{cases} 0 & \text{if } x<0 \\ \pi/2 & \text{if } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$$

2. $$\int_0^\infty \frac{\sin \pi w \sin xw}{1-w^2}\,dw = \begin{cases} \frac{\pi}{2}\sin x & \text{if } 0 \leqq x \leqq \pi \\ 0 & \text{if } x>\pi \end{cases}$$

3. $$\int_0^\infty \frac{1-\cos \pi w}{w}\sin xw\,dw = \begin{cases} \frac{1}{2}\pi & \text{if } 0<x<\pi \\ 0 & \text{if } x>\pi \end{cases}$$

4. $$\int_0^\infty \frac{\cos \frac{1}{2}\pi w}{1-w^2}\cos xw\,dw = \begin{cases} \frac{1}{2}\pi\cos x & \text{if } 0<|x|<\frac{1}{2}\pi \\ 0 & \text{if } |x| \geqq \frac{1}{2}\pi \end{cases}$$

5. $$\int_0^\infty \frac{\sin w - w\cos w}{w^2}\sin xw\,dw = \begin{cases} \frac{1}{2}\pi x & \text{if } 0<x<1 \\ \frac{1}{4}\pi & \text{if } x=1 \\ 0 & \text{if } x>1 \end{cases}$$

6. $$\int_0^\infty \frac{w^3\sin xw}{w^4+4}\,dw = \tfrac{1}{2}\pi e^{-x}\cos x \quad \text{if } x>0$$

7–12 FOURIER COSINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (10).

7. $f(x) = \begin{cases} 1 & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$

8. $f(x) = \begin{cases} x^2 & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$

9. $f(x) = 1/(1+x^2)$ $[x>0$. *Hint*. See (13).]

10. $f(x) = \begin{cases} a^2-x^2 & \text{if } 0<x<a \\ 0 & \text{if } x>a \end{cases}$

11. $f(x) = \begin{cases} \sin x & \text{if } 0<x<\pi \\ 0 & \text{if } x>\pi \end{cases}$

12. $f(x) = \begin{cases} e^{-x} & \text{if } 0<x<a \\ 0 & \text{if } x>a \end{cases}$

13. CAS EXPERIMENT. Approximate Fourier Cosine Integrals. Graph the integrals in Prob. 7, 9, and 11 as functions of x. Graph approximations obtained by replacing ∞ with finite upper limits of your choice. Compare the quality of the approximations. Write a short report on your empirical results and observations.

14. PROJECT. Properties of Fourier Integrals

(a) Fourier cosine integral. Show that (10) implies

(a1) $$f(ax) = \frac{1}{a}\int_0^\infty A\left(\frac{w}{a}\right)\cos xw\,dw$$

$(a>0)$ (*Scale change*)

(a2) $$xf(x) = \int_0^\infty B^*(w)\sin xw\,dw,$$

$$B^* = -\frac{dA}{dw}, \qquad A \text{ as in (10)}$$

(a3) $$x^2 f(x) = \int_0^\infty A^*(w)\cos xw\,dw,$$

$$A^* = -\frac{d^2A}{dw^2}.$$

(b) Solve Prob. 8 by applying (a3) to the result of Prob. 7.

(c) Verify (a2) for $f(x)=1$ if $0<x<a$ and $f(x)=0$ if $x>a$.

(d) Fourier sine integral. Find formulas for the Fourier sine integral similar to those in (a).

15. CAS EXPERIMENT. Sine Integral. Plot $\text{Si}(u)$ for positive u. Does the sequence of the maximum and minimum values give the impression that it converges and has the limit $\pi/2$? Investigate the Gibbs phenomenon graphically.

16–20 FOURIER SINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (11).

16. $f(x) = \begin{cases} x & \text{if } 0<x<a \\ 0 & \text{if } x>a \end{cases}$

17. $f(x) = \begin{cases} 1 & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$

18. $f(x) = \begin{cases} \cos x & \text{if } 0<x<\pi \\ 0 & \text{if } x>\pi \end{cases}$

19. $f(x) = \begin{cases} e^{x} & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$

20. $f(x) = \begin{cases} e^{-x} & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$

PROBLEM SET 11.8

1–8 FOURIER COSINE TRANSFORM

1. Find the cosine transform $\hat{f}_c(w)$ of $f(x) = 1$ if $0 < x < 1$, $f(x) = -1$ if $1 < x < 2$, $f(x) = 0$ if $x > 2$.

2. Find f in Prob. 1 from the answer $\hat{f}_c$.

3. Find $\hat{f}_c(w)$ for $f(x) = x$ if $0 < x < 2$, $f(x) = 0$ if $x > 2$.

4. Derive formula 3 in Table I of Sec. 11.10 by integration.

5. Find $\hat{f}_c(w)$ for $f(x) = x^2$ if $0 < x < 1$, $f(x) = 0$ if $x > 1$.

6. Continuity assumptions. Find $\hat{g}_c(w)$ for $g(x) = 2$ if $0 < x < 1$, $g(x) = 0$ if $x > 1$. Try to obtain from it $\hat{f}_c(w)$ for $f(x)$ in Prob. 5 by using (5a).

7. Existence? Does the Fourier cosine transform of $x^{-1} \sin x$ $(0 < x < \infty)$ exist? Of $x^{-1} \cos x$? Give reasons.

8. Existence? Does the Fourier cosine transform of $f(x) = k = \text{const}$ $(0 < x < \infty)$ exist? The Fourier sine transform?

9–15 FOURIER SINE TRANSFORM

9. Find $\mathscr{F}_s(e^{-ax})$, $a > 0$, by integration.

10. Obtain the answer to Prob. 9 from (5b).

11. Find $f_s(w)$ for $f(x) = x^2$ if $0 < x < 1$, $f(x) = 0$ if $x > 1$.

12. Find $\mathscr{F}_s(xe^{-x^2/2})$ from (4b) and a suitable formula in Table I of Sec. 11.10.

13. Find $\mathscr{F}_s(e^{-x})$ from (4a) and formula 3 of Table I in Sec. 11.10.

14. Gamma function. Using formulas 2 and 4 in Table II of Sec. 11.10, prove $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ [(30) in App. A3.1], a value needed for Bessel functions and other applications.

15. WRITING PROJECT. Finding Fourier Cosine and Sine Transforms. Write a short report on ways of obtaining these transforms, with illustrations by examples of your own.

11.9 Fourier Transform. Discrete and Fast Fourier Transforms

In Sec. 11.8 we derived two real transforms. Now we want to derive a complex transform that is called the **Fourier transform**. It will be obtained from the complex Fourier integral, which will be discussed next.

Complex Form of the Fourier Integral

The (real) Fourier integral is [see (4), (5), Sec. 11.7]

$$f(x) = \int_0^\infty [A(w) \cos wx + B(w) \sin wx] \, dw$$

where

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv, \qquad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv \, dv.$$

Substituting A and B into the integral for f, we have

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} f(v)[\cos wv \cos wx + \sin wv \sin wx] \, dv \, dw.$$

PROBLEM SET 11.9

1. **Review in complex.** Show that $1/i = -i$, $e^{-ix} = \cos x - i \sin x$, $e^{ix} + e^{-ix} = 2 \cos x$, $e^{ix} - e^{-ix} = 2i \sin x$, $e^{ikx} = \cos kx + i \sin kx$.

2–11 FOURIER TRANSFORMS BY INTEGRATION

Find the Fourier transform of $f(x)$ (without using Table III in Sec. 11.10). Show details.

2. $f(x) = \begin{cases} e^{2ix} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

3. $f(x) = \begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$

4. $f(x) = \begin{cases} e^{kx} & \text{if } x < 0 \quad (k > 0) \\ 0 & \text{if } x > 0 \end{cases}$

5. $f(x) = \begin{cases} e^{x} & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$

6. $f(x) = e^{-|x|} \quad (-\infty < x < \infty)$

7. $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$

8. $f(x) = \begin{cases} xe^{-x} & \text{if } -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$

9. $f(x) = \begin{cases} |x| & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

10. $f(x) = \begin{cases} x & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

11. $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

12–17 USE OF TABLE III IN SEC. 11.10. OTHER METHODS

12. Find $\mathscr{F}(f(x))$ for $f(x) = xe^{-x}$ if $x > 0$, $f(x) = 0$ if $x < 0$, by (9) in the text and formula 5 in Table III (with $a = 1$). *Hint.* Consider xe^{-x} and e^{-x}.

13. Obtain $\mathscr{F}(e^{-x^2/2})$ from Table III.

14. In Table III obtain formula 7 from formula 8.

15. In Table III obtain formula 1 from formula 2.

16. **TEAM PROJECT. Shifting (a)** Show that if $f(x)$ has a Fourier transform, so does $f(x - a)$, and $\mathscr{F}\{f(x - a)\} = e^{-iwa}\mathscr{F}\{f(x)\}$.

    (b) Using (a), obtain formula 1 in Table III, Sec. 11.10, from formula 2.

    (c) Shifting on the *w*-Axis. Show that if $\hat{f}(w)$ is the Fourier transform of $f(x)$, then $\hat{f}(w - a)$ is the Fourier transform of $e^{iax}f(x)$.

    (d) Using (c), obtain formula 7 in Table III from 1 and formula 8 from 2.

17. What could give you the idea to solve Prob. 11 by using the solution of Prob. 9 and formula (9) in the text? Would this work?

18–25 DISCRETE FOURIER TRANSFORM

18. Verify the calculations in Example 4 of the text.

19. Find the transform of a general signal $f = [f_1 \quad f_2 \quad f_3 \quad f_4]^{\mathsf{T}}$ of four values.

20. Find the inverse matrix in Example 4 of the text and use it to recover the given signal.

21. Find the transform (the frequency spectrum) of a general signal of two values $[f_1 \quad f_2]^{\mathsf{T}}$.

22. Recreate the given signal in Prob. 21 from the frequency spectrum obtained.

23. Show that for a signal of eight sample values, $w = e^{-i/4} = (1 - i)/\sqrt{2}$. Check by squaring.

24. Write the Fourier matrix **F** for a sample of eight values explicitly.

25. **CAS Problem.** Calculate the inverse of the 8×8 Fourier matrix. Transform a general sample of eight values and transform it back to the given data.

CHAPTER 11 REVIEW QUESTIONS AND PROBLEMS

1. What is a Fourier series? A Fourier cosine series? A half-range expansion? Answer from memory.
2. What are the Euler formulas? By what very important idea did we obtain them?
3. How did we proceed from 2π-periodic to general-periodic functions?
4. Can a discontinuous function have a Fourier series? A Taylor series? Why are such functions of interest to the engineer?
5. What do you know about convergence of a Fourier series? About the Gibbs phenomenon?
6. The output of an ODE can oscillate several times as fast as the input. How come?
7. What is approximation by trigonometric polynomials? What is the minimum square error?
8. What is a Fourier integral? A Fourier sine integral? Give simple examples.
9. What is the Fourier transform? The discrete Fourier transform?
10. What are Sturm–Liouville problems? By what idea are they related to Fourier series?

11–20 FOURIER SERIES. In Probs. 11, 13, 16, 20 find the Fourier series of $f(x)$ as given over one period and sketch $f(x)$ and partial sums. In Probs. 12, 14, 15, 17–19 give answers, with reasons. Show your work detail.

11. $f(x) = \begin{cases} 0 & \text{if} \quad -2 < x < 0 \\ 2 & \text{if} \quad 0 < x < 2 \end{cases}$
12. Why does the series in Prob. 11 have no cosine terms?
13. $f(x) = \begin{cases} 0 & \text{if} \quad -1 < x < 0 \\ x & \text{if} \quad 0 < x < 1 \end{cases}$
14. What function does the series of the cosine terms in Prob. 13 represent? The series of the sine terms?
15. What function do the series of the cosine terms and the series of the sine terms in the Fourier series of e^x $(-5 < x < 5)$ represent?
16. $f(x) = |x| \quad (-\pi < x < \pi)$
17. Find a Fourier series from which you can conclude that $1 - 1/3 + 1/5 - 1/7 + - \cdots = \pi/4$.
18. What function and series do you obtain in Prob. 16 by (termwise) differentiation?
19. Find the half-range expansions of $f(x) = x$ $(0 < x < 1)$.
20. $f(x) = 3x^2 \quad (-\pi < x < \pi)$

21–22 GENERAL SOLUTION

Solve, $y'' + \omega^2 y = r(t)$, where $|\omega| \neq 0, 1, 2, \cdots, r(t)$ is 2π-periodic and

21. $r(t) = 3t^2 \; (-\pi < t < \pi)$
22. $r(t) = |t| \; (-\pi < t < \pi)$

23–25 MINIMUM SQUARE ERROR

23. Compute the minimum square error for $f(x) = x/\pi$ $(-\pi < x < \pi)$ and trigonometric polynomials of degree $N = 1, \cdots, 5$.
24. How does the minimum square error change if you multiply $f(x)$ by a constant k?
25. Same task as in Prob. 23, for $f(x) = |x|/\pi$ $(-\pi < x < \pi)$. Why is E^* now much smaller (by a factor 100, approximately!)?

26–30 FOURIER INTEGRALS AND TRANSFORMS

Sketch the given function and represent it as indicated. If you have a CAS, graph approximate curves obtained by replacing ∞ with finite limits; also look for Gibbs phenomena.

26. $f(x) = x + 1$ if $0 < x < 1$ and 0 otherwise; by the Fourier sine transform
27. $f(x) = x$ if $0 < x < 1$ and 0 otherwise; by the Fourier integral
28. $f(x) = kx$ if $a < x < b$ and 0 otherwise; by the Fourier transform
29. $f(x) = x$ if $1 < x < a$ and 0 otherwise; by the Fourier cosine transform
30. $f(x) = e^{-2x}$ if $x > 0$ and 0 otherwise; by the Fourier transform