EE-232 Signals & Systems

Lecture 10

Sampling Discrete-Time Systems

Asst Prof Kamran Aziz Bhatti

Lecture 10: Sampling Discrete-Time Systems

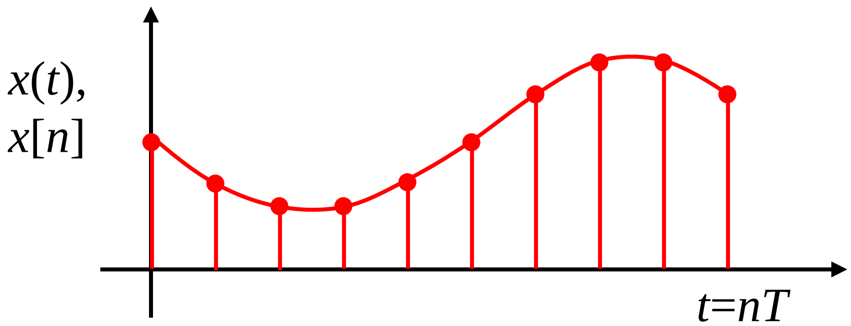
Sampling & Discrete-time systems (2 lectures):
Sampling theorem, discrete Fourier transform

Specific objectives for today:

- **Sampling** of a continuous-time signal
- **Reconstruction** of the signals from its samples
- **Sampling theorem** & **Nyquist rate**
- Reconstruction of a signal, using **zero-order holds**

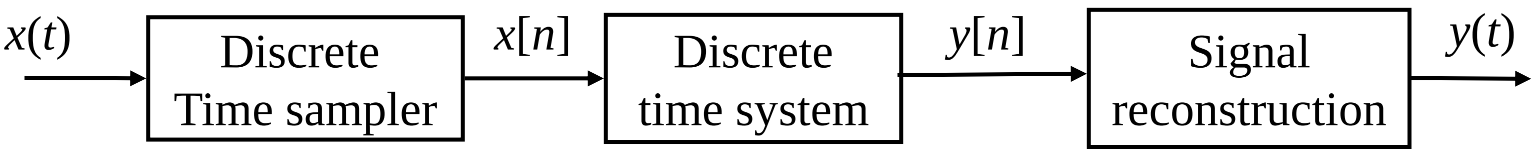
What is Discrete Time Sampling?

Sampling is the transformation of a continuous signal into a discrete signal



T is the sampling period

Widely applied in digital analysis systems



1. Sample the continuous time signal
2. Design and process discrete time signal
3. Convert back to continuous time

Why is Sampling Important?

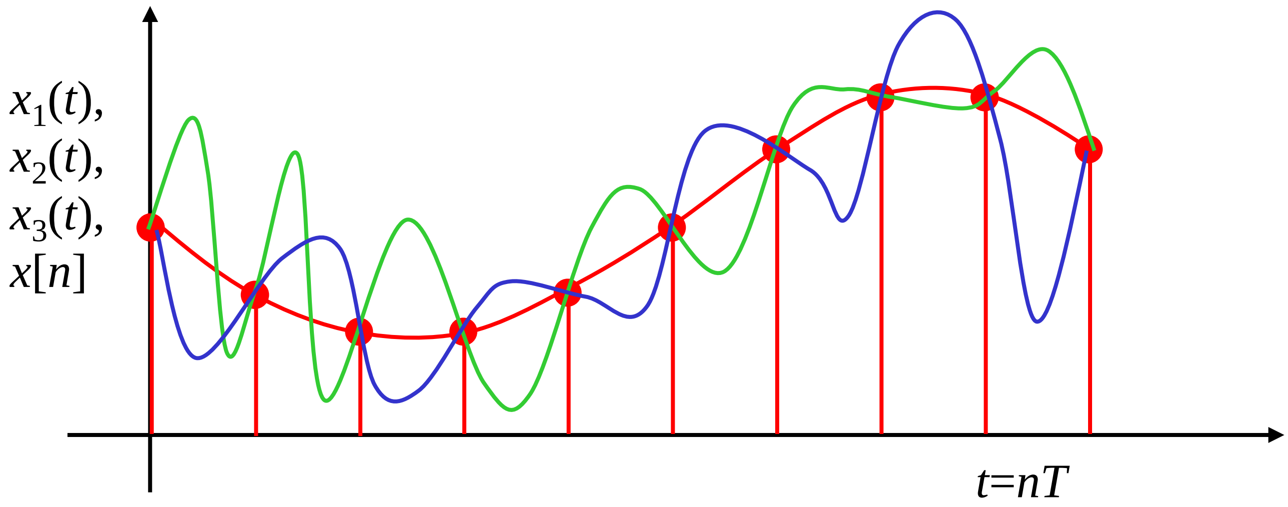
For many systems (e.g. Matlab, …) designing and processing discrete-time systems is more efficient and more general compared to performing continuous-time system design.

- How does Simulink perform continuous-time system simulation?
- The signals are sampled and the systems are approximately integrated in discrete time

Mainly due to the dramatic development of digital technology resulting in inexpensive, lightweight, programmable and reproducible **discrete-time** systems. Widely used for communication

Sampling a Continuous-Time Signal

Clearly for a finite sample period *T*, it is not possible to represent every uncountable, infinite-dimensional continuous-time signal with a countable, infinite-dimensional discrete-time signal.



In general, an infinite number of CT signals can generate a DT signal.

However, if the signal is band (frequency) limited, and the samples are sufficiently close, it is possible to uniquely reconstruct the original CT signal from the sampled signal

Definition of Impulse Train Sampling

We need to have a convenient way in which to represent the sampling of a CT signal at regular intervals

A common/useful way to do this is through the use of a periodic **impulse train** signal, $p(t)$, multiplied by the CT signal

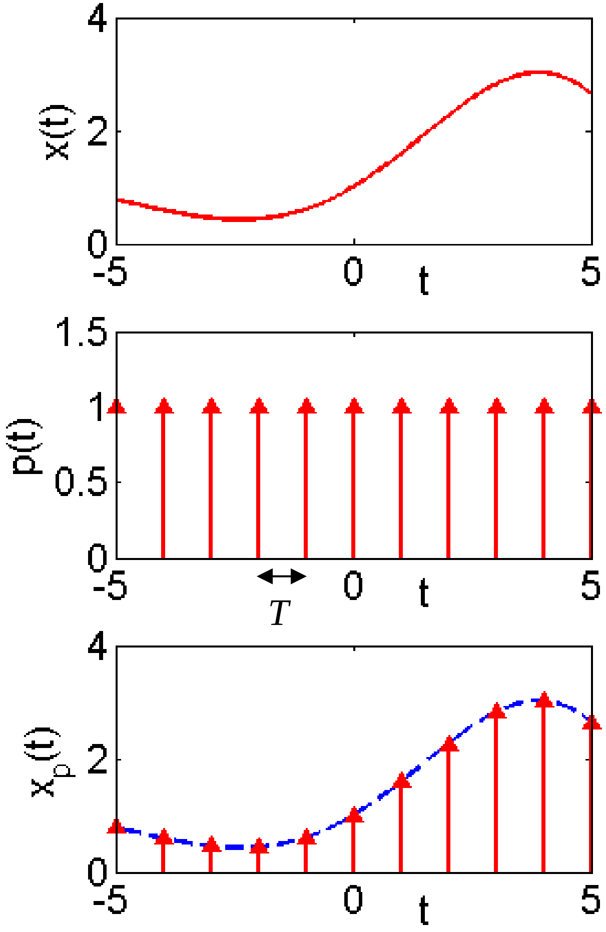
$$x_p(t) = x(t)\,p(t)$$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

T is the sampling period

$\omega_s = 2\pi/T$ is the sampling frequency

This is known as **impulse train sampling**. Note $x_p(t)$ is still a continuous time signal



Analysing Impulse Train Sampling (i)

What effect does this sampling have on the frequency decomposition (Fourier transform) of the CT impulse train signal $x_p(t)$?

By definition:

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

The signal $p(t)$ is **periodic** and the **coefficients of the Fourier Series** are given by:

$$a_k = \tfrac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt = \tfrac{1}{T}, \qquad \forall k$$

Therefore, the Fourier transform is given by

$$P(j\omega) = \tfrac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

One property of the Fourier transform we did not consider is the **multiplicative property** which says if $x_p(t) = x(t)p(t)$, then

$$X_p(j\omega) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))d\theta$$

Analysing Impulse Train Sampling (ii)

Substituting for $P(j\omega)$

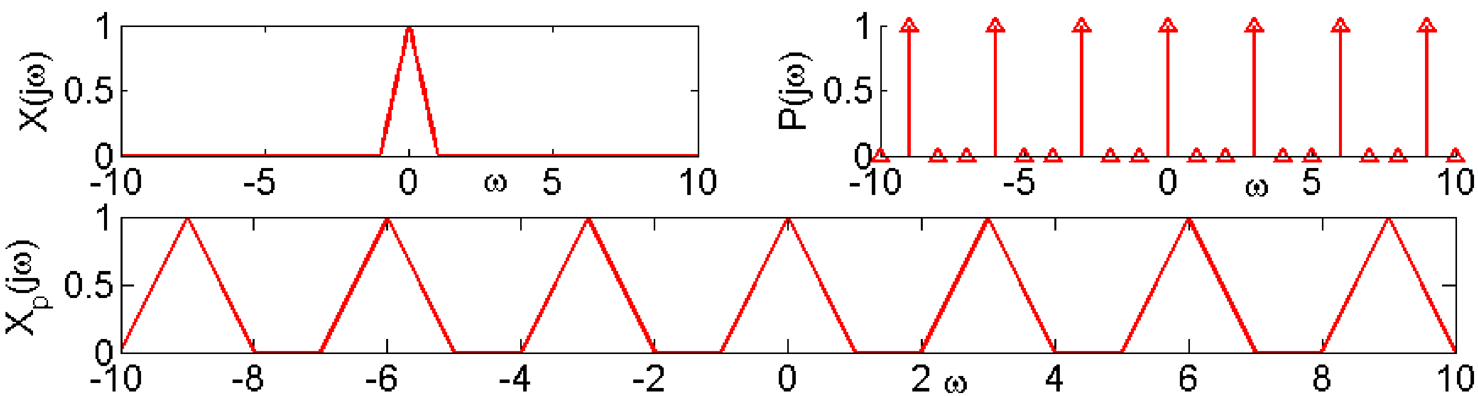
$$X_p(j\omega) = \frac{1}{T}\int_{-\infty}^{\infty} X(j\theta)\sum_{k=-\infty}^{\infty}\delta((\omega - k\omega_s) - \theta)d\theta$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} X(j\theta)\delta((\omega - k\omega_s) - \theta)d\theta$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

Therefore $X_p(j\omega)$ is a periodic function of ω, consisting of a superposition of shifted replicas of $X(j\omega)$, scaled by $1/T$.

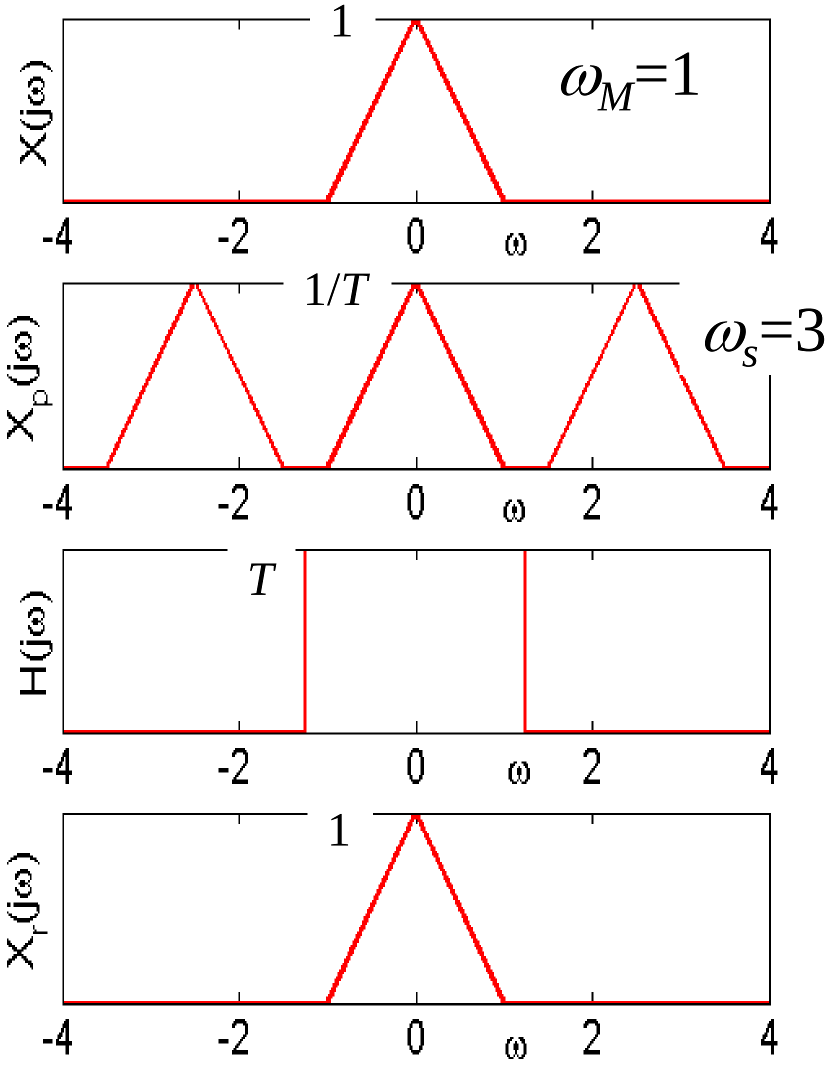
$|X(j\omega)|=0$: $|\omega|>1$ $\qquad$ $\omega_s=3$



Reconstruction of the CT Signal

When the sampling frequency ω_s is less than twice the band-limited frequency ω_M, there is no overlaps the spectrum X($j\omega$)

If this is true, the original signal $x(t)$ can be recovered from the impulse sampled $x_p(t)$, by passing it through a low pass filter $H(j\omega)$ with gain T and cutoff frequency between ω_M and ω_s-ω_M.



Sampling Theorem

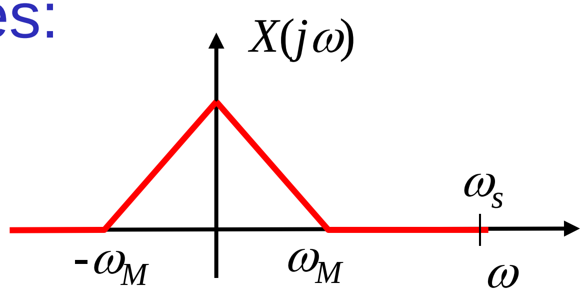
Let $x(t)$ be a **band (frequency)-limited signal**

$X(j\omega) = 0$ for $|\omega| > \omega_M$.

Then $x(t)$ is **uniquely determined by its samples** $\{x(nT)\}$ when the sampling frequency satisfies:

$$\omega_s > 2\omega_M$$

where $\omega_s = 2\pi/T$.



$2\omega_M$ is known as the **Nyquist rate**, as it represents the largest frequency that can be reproduced with the sample time

The result makes sense because a frequency-limited signal has a limited amount of information that can be fully captured with the sampled sequence $\{x(nT)\}$

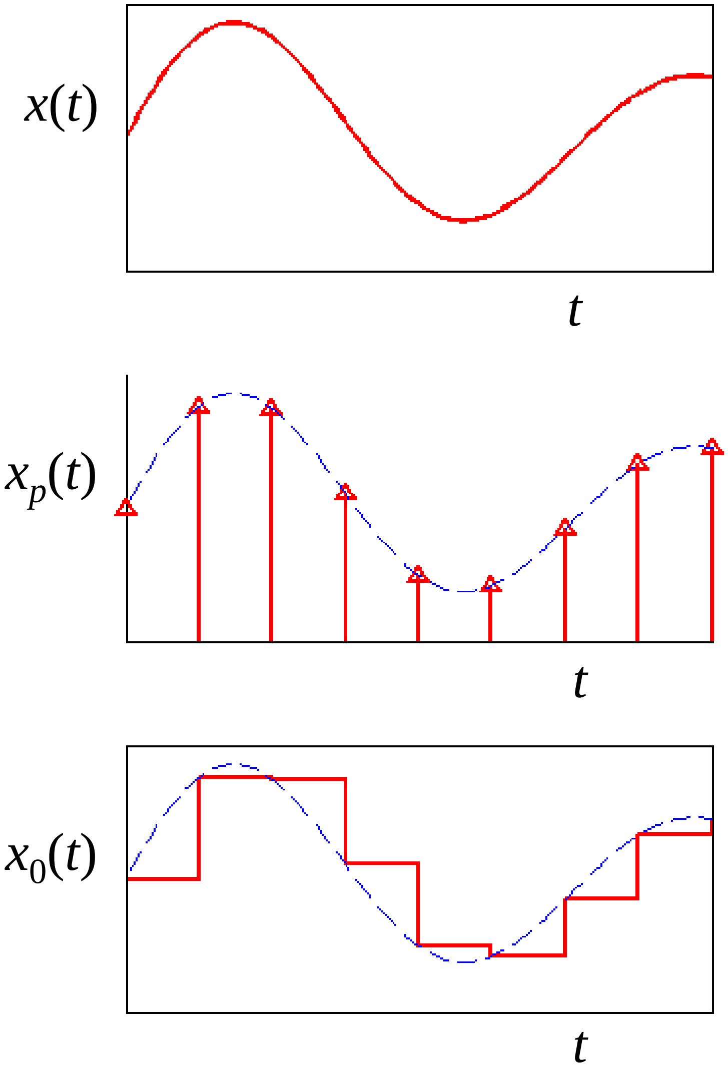
Zero Order Hold Sampling

A zero order hold is a common method for bridging CT-DT signals

A zero order hold samples the current signal and holds that value until the next sample

In most systems, it is difficult to generate and transmit narrow, large-amplitude pulses (impulse train sampling) $x_p(t)$

We can often use a variety of filtering/interpolation techniques to reconstruct the original time-domain signal, however often the zero-order hold signal $x_0(t)$ is sufficiently accurate



Lecture 10: Summary

The sample time for converting a continuous time signal into a sampled, discrete time signal is determined by the **Nyquist rate**, amongst other things.

The signal must satisfy the relationship:

$$\omega_s > 2\omega_M$$

If the signal is to be preserved exactly. Information in frequencies higher than this will be lost when the signal is sampled.

A continuous time signal is often sampled and communicated using a **zero order hold**

Often this is enough to be considered as the re-constructed continuous time signal, but sometimes approximate methods for re-constructing the signal are used

Lecture 10: Exercises

Theory

SaS, 7.1-7.4, 7.7