EE-232 Signals & Systems

Lecture 11

Discrete Fourier Transform

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Lecture 11: Discrete Fourier Transform

Sampling Discrete-time systems (2 lectures): Sampling theorem, **discrete Fourier transform**

Specific objectives for today:

- Discrete Fourier transform
- Examples
- Convergence & properties
- Convolution

Reminder: CT Fourier Transform

CT Fourier transform maps a time domain frequency signal to the frequency domain via

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

The Fourier transform is used to

- Analyse the frequency content of a signal
- Design a system/filter with particular properties
- Solve differential equations in the frequency domain using algebraic operators

Note that the transform/integral is not defined for some signals (infinite energy)

Derivation of the DT Fourier Transform

By analogy with the CT Fourier transform, we might "guess"

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

This is because $t \approx n$, & the integral operator represents the limit of a sum as the sum's width tends to zero.

$X(e^{j\omega})$ is periodic of period 2π, so is $e^{j\omega n}$. Try substituting into the inverse Fourier transform with integral over 2π:

$$\begin{aligned} \bar{x}[n] &= \tfrac{1}{2\pi} \int_{2\pi} \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} e^{j\omega n} d\omega \\ &= \tfrac{1}{2\pi} \sum_{k=-\infty}^{\infty} x[k] \int_{2\pi} e^{j\omega(n-k)} d\omega \\ &= x[n] \end{aligned}$$

which is the original DT signal.

Discrete Time Fourier Transform

The DT Fourier transform **analysis** and **synthesis** equations are therefore:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

The function $X(e^{j\omega})$ is referred to as the **discrete-time Fourier transform** and the pair of equations are referred to as the **Fourier transform pair**

$X(e^{j\omega})$ is sometimes referred to as the **spectrum** of $x[n]$ because it provides us with information on how $x[n]$ is composed of **complex exponentials at different frequencies**

It **converges** when the signal is **absolutely summable**

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Example 1: 1st Order System, Decay Power

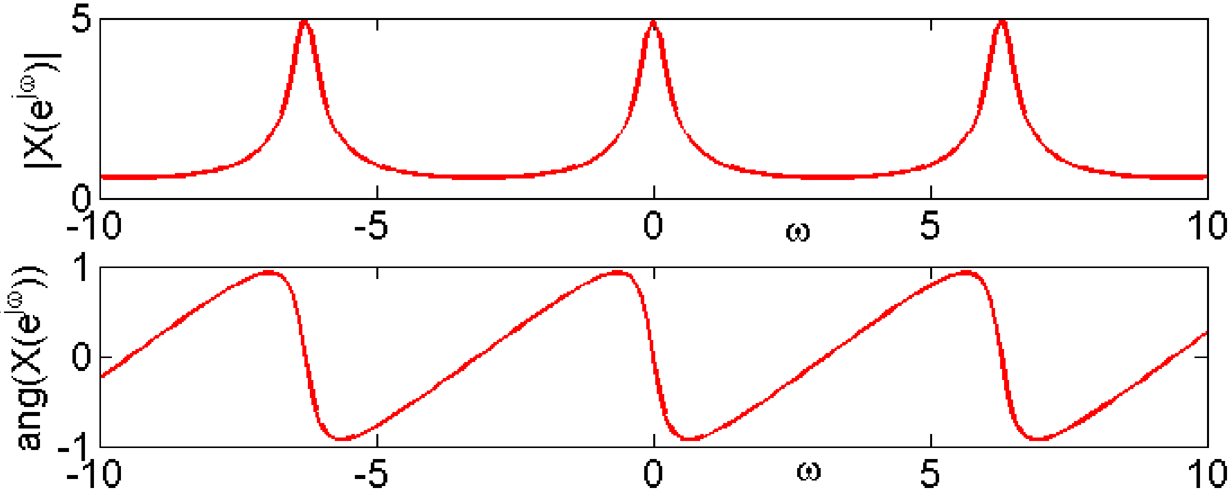
Calculate the DT Fourier transform of the signal:

$$x[n] = a^n u[n], \qquad |a| < 1 \qquad \text{stable system}$$

Therefore:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n = \frac{1}{1 - a e^{-j\omega}}$$

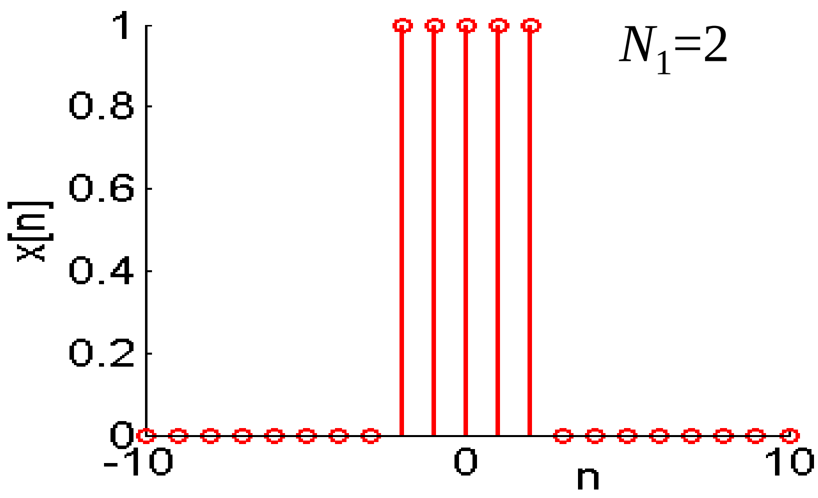


a=0.8

Example 2: Rectangular Pulse

Consider the rectangular pulse

$$x[n] = \begin{cases} 1 & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases}$$



and the Fourier transform is

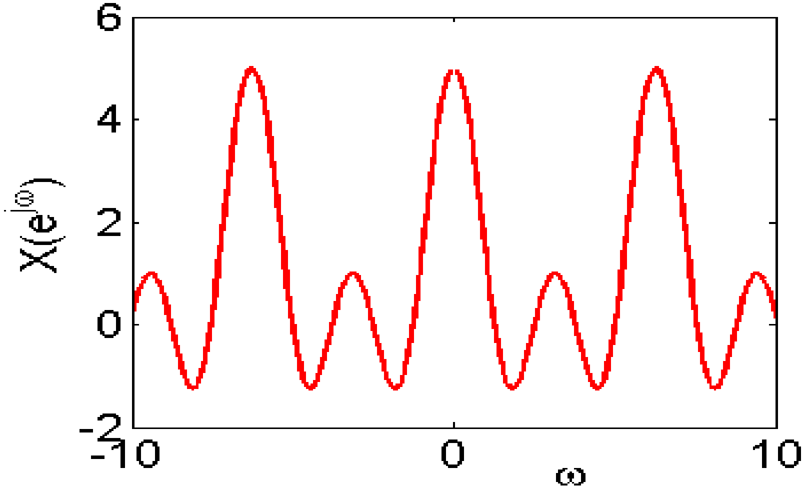
$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

$$= e^{j\omega N_1} \sum_{m=0}^{2N_1} e^{-j\omega m}$$

$$= \frac{e^{j\omega(N_1+1/2)}}{e^{j\omega(1/2)}} \frac{1-e^{-j\omega(2N_1+1)}}{1-e^{-j\omega}}$$

$$= \frac{e^{j\omega(N_1+1/2)} - e^{-j\omega(N_1+1/2)}}{e^{j\omega/2} - e^{-j\omega/2}}$$

$$= \frac{\sin(\omega(N_1+1/2))}{\sin(\omega/2)}$$



Example 3: Impulse Signal

Fourier transform of the DT impulse signal is

$$F\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$$

Properties: Periodicity, Linearity & Time

The DT Fourier transform is always **periodic with period 2π**, because

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

It is relatively straightforward to prove that the DT Fourier transform is **linear**, i.e.

$$x_1[n] \overset{F}{\leftrightarrow} X_1(e^{j\omega}), \quad x_2[n] \overset{F}{\leftrightarrow} X_2(e^{j\omega})$$

$$ax_1[n] + bx_2[n] \overset{F}{\leftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

Similarly, if a DT signal is **shifted** by n_0 units of time

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$$

$$x[n-n_0] \overset{F}{\leftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$

Convolution in the Frequency Domain

Like continuous time signals and systems, the time-domain convolution of two discrete time signals can be represented as the **multiplication** of the Fourier transforms

If $x[n]$, $h[n]$ and $y[n]$ are the input, **impulse response** and output of a discrete-time LTI system so, by convolution,

$$y[n] = x[n]*h[n]$$

Then

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

The proof is analogous to proof used for the convolution of continuous time Fourier transforms

Convolution in the discrete time domain is replaced by multiplication in the frequency domain.

Example: 1st Order System

Consider an LTI system with impulse response

$h[n]=\alpha^n u[n], \qquad |\alpha|<1$

and the system input is

$x[n]=\beta^n u[n], \qquad |\beta|<1$

The DT Fourier transforms are:

$$H(e^{j\omega})=\frac{1}{1-\alpha e^{-j\omega}}, \qquad X(e^{j\omega})=\frac{1}{1-\beta e^{-j\omega}}$$

So

$$Y(e^{j\omega})=\frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}$$

Expressing as partial fractions, assuming $\alpha\neq\beta$:

$$Y(e^{j\omega})=\frac{\alpha}{\alpha-\beta}\frac{1}{(1-\alpha e^{-j\omega})}-\frac{\beta}{\alpha-\beta}\frac{1}{(1-\beta e^{-j\omega})}$$

and spotting the inverse Fourier transform

$$y[n]=\frac{\alpha^{n+1}u[n]}{\alpha-\beta}-\frac{\beta^{n+1}u[n]}{\alpha-\beta}$$

Lecture 11: Summary

Apart from a slightly difference, the Fourier transform of a discrete time signal is equivalent to the continuous time formulae

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

They have similar properties to the continuous time Fourier transform for linearity, time shifts, differencing and accumulation

The main result is that like continuous time signals and systems, convolution in the time domain is replaced by multiplication in the frequency domain.

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Lecture 11: Exercises

Theory

SaS, 5.1-5.3, 5.19