EE-232 Signals & Systems

Lecture 13

Inverse Laplace Transform

Asst Prof Kamran Aziz Bhatti

Lecture 13: Inverse Laplace Transform

Laplace transform (3 lectures):

Laplace transform as Fourier transform with convergence factor. Properties of the Laplace transform

Specific objectives for today:

- Poles and zeros of a Laplace transfer function
- Rational polynomial transfer functions
- Inverse Laplace transform

Reminder: Laplace Transforms

Equivalent to the Fourier transform when *s*=$j\omega$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$x(t) \overset{L}{\leftrightarrow} X(s)$$

There is an associated region of convergence for *s* where the (transformed) signal has finite energy. The Laplace transform is only defined for these values

Laplace transform is linear (easy!)

Examples for the Laplace transforms include

$$e^{-at}u(t) \overset{L}{\leftrightarrow} \frac{1}{s+a}, \quad \mathrm{Re}\{s\} > -a$$

$$3e^{-2t}u(t) - 2e^{-t}u(t) \overset{L}{\leftrightarrow} \frac{s-1}{s^2+3s+2}, \qquad \mathrm{Re}\{s\} > -1$$

Ratio of Polynomials

In each of these examples, the Laplace transform is rational, i.e. it is a ratio of polynomials in the complex variable *s*.

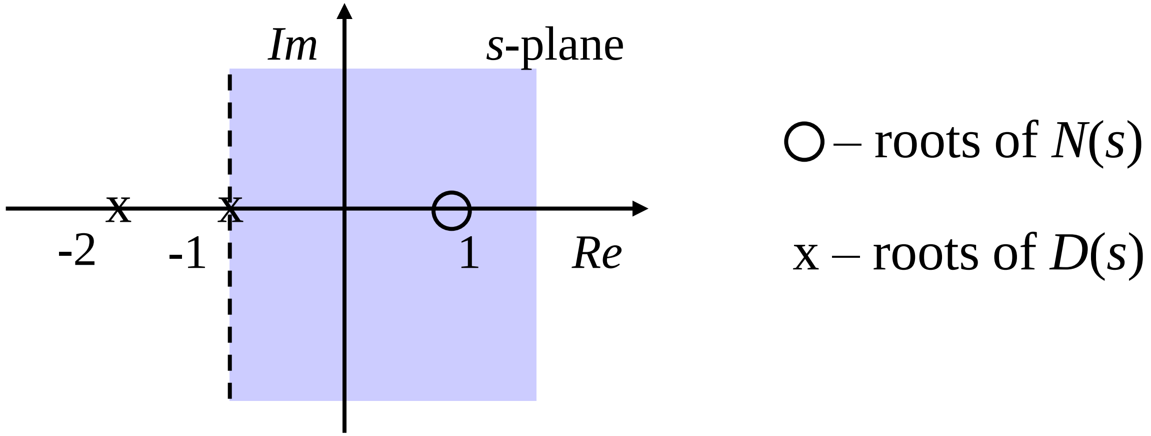
$$X(s) = \frac{N(s)}{D(s)}$$

where *N* and *D* are the numerator and denominator polynomial respectively.

In fact, *X*(*s*) will be rational whenever *x*(*t*) is a linear combination of real or complex exponentials. **Rational transforms** also arise when we consider **LTI systems specified in terms of linear, constant coefficient differential equations**.

We can mark the roots of *N* and *D* in the *s*-plane along with the ROC

Example 3:



Poles and Zeros

The roots of $N(s)$ are known as the **zeros**. For these values of s, $X(s)$ is zero.

The roots of $D(s)$ are known as the **poles**. For these values of s, $X(s)$ is infinite, the Region of Convergence for the Laplace transform cannot contain any poles, because the corresponding integral is infinite

The set of poles and zeros completely characterise $X(s)$ to within a scale factor (+ ROC for Laplace transform)

$$X(s) \propto \frac{\prod_i (s - z_i)}{\prod_j (s - p_j)}$$

The graphical representation of $X(s)$ through its poles and zeros in the s-plane is referred to as the **pole-zero** plot of $X(s)$

Example: Poles and Zeros

Consider the signal:

$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$$

By linearity (& last lecture) we can evaluate the second and third terms

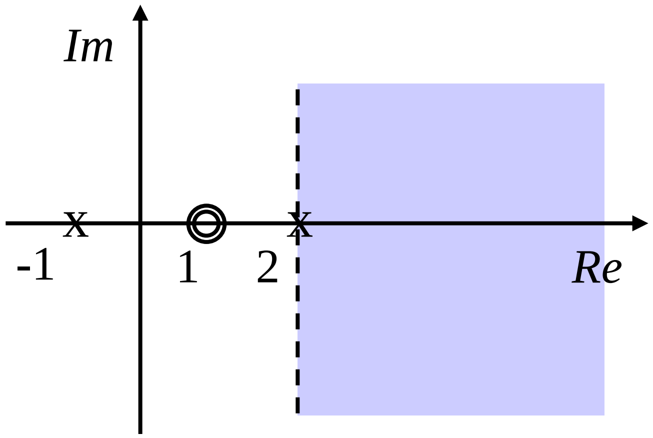
The Laplace transform of the impulse function is:

$$L\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt = 1$$

which is valid for any *s*. Therefore,

$$X(s) = 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2}$$

$$= \frac{(s-1)^2}{(s+1)(s-2)}, \qquad \text{Re}\{s\} > 2$$



ROC Properties for Laplace Transform

Property 1: The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane

Because the Laplace transform consists of s for which $x(t)e^{-\sigma t}$ converges, which only depends on $\text{Re}\{s\} = \sigma$

Property 2: For rational Laplace transforms, the ROC does not contain any poles

Because $X(s)$ is infinite at a pole, the integral must not converge.

Property 3: if $x(t)$ is finite duration and is absolutely integrable then the ROC is the entire s-plane.

Because $x(t)$ is magnitude bounded, multiplication by any exponential over a finite interval is also bounded. Therefore the Laplace integral converges for any s.

Inverse Laplace Transform

The Laplace transform of a signal *x*(*t*) is:

$$X(\sigma + j\omega) = F\{x(t)e^{-\sigma t}\} = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt$$

We can invert this relationship using the inverse Fourier transform

$$x(t)e^{-\sigma t} = F^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

Multiplying both sides by $e^{\sigma t}$:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega$$

Therefore, we can recover *x*(*t*) from *X*(*s*), where the real component is fixed and we integrate over the imaginary part, noting that $ds = jd\omega$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

Inverse Laplace Transform Interpretation

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

- Just about all real-valued signals, $x(t)$, can be represented as a weighted, $X(s)$, integral of complex exponentials, e^{st}.
- The contour of integration is a straight line (in the complex plane) from σ-$j\infty$ to σ+$j\infty$ (**we won't be explicitly evaluating this, just spotting known transformations**)
- We can choose any σ for this integration line, as long as the integral converges
- For the class of **rational Laplace transforms**, we can express $X(s)$ as **partial fractions** to determine the **inverse Fourier transform**.

$$X(s) = \sum_{i=1}^{M} \frac{A_i}{s + a_i}$$

$$L^{-1}\{A_i/(s+a_i)\} \to \begin{cases} A_i e^{-a_i t}u(t) & \mathrm{Re}\{s\} > -a_i \\ -A_i e^{-a_i t}u(-t) & \mathrm{Re}\{s\} < -a_i \end{cases}$$

Example 1: Inverting the Laplace Transform

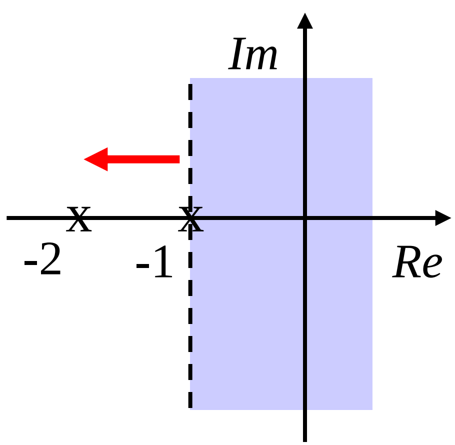
Consider when

$$X(s) = \frac{1}{(s+1)(s+2)} \qquad \Re(s) > -1$$

Like the inverse Fourier transform, expand as partial fractions

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} = \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

Pole-zero plots and ROC for combined & individual terms



$$e^{-t}u(t) \overset{L}{\leftrightarrow} \frac{1}{s+1}, \qquad \mathrm{Re}\{s\} > -1$$

$$e^{-2t}u(t) \overset{L}{\leftrightarrow} \frac{1}{s+2}, \qquad \mathrm{Re}\{s\} > -2$$

$$x(t) = (e^{-t} - e^{-2t})u(t) \overset{L}{\leftrightarrow} \frac{1}{(s+1)(s+2)}, \qquad \mathrm{Re}\{s\} > -1$$

Example 2

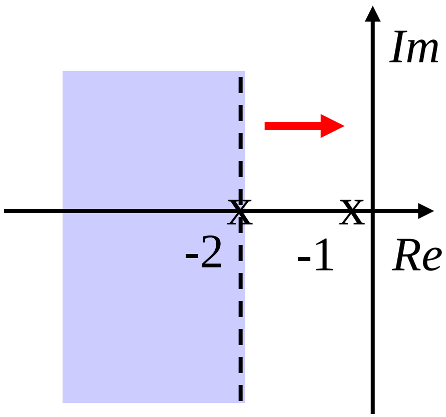
Consider when

$$X(s) = \frac{1}{(s+1)(s+2)} \qquad \mathrm{Re}\{s\} < -2$$

Like the inverse Fourier transform, expand as partial fractions

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} = \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

Pole-zero plots and ROC for combined & individual terms



$$-e^{-t}u(-t) \overset{L}{\leftrightarrow} \frac{1}{s+1}, \qquad \mathrm{Re}\{s\} < -1$$

$$-e^{-2t}u(-t) \overset{L}{\leftrightarrow} \frac{1}{s+2}, \qquad \mathrm{Re}\{s\} < -2$$

$$x(t) = (-e^{-t} + e^{-2t})u(-t) \overset{L}{\leftrightarrow} \frac{1}{(s+1)(s+2)}, \qquad \mathrm{Re}\{s\} < -2$$

Lecture 13: Summary

For many signals that are made up of a linear combination of complex exponentials and CT LTI systems that are described by differential equations, the Laplace transform is rational, i.e. it is a ratio of polynomials in *s*: $N(s)/D(s)$

The roots of $N(s)$ and $D(s)$ are known as the **zeros and poles** of the transfer function, respectively.

The Region of Convergence does not contain any poles

The inverse Laplace transform is given by

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

It is usually calculated by expressing the Laplace transform as partial fractions, and then spotting known relationships (rather than directly evaluating the inverse transform)

Questions

Theory

SaS, O&W, Q9.9, 9.22. Also, prove $\delta(t) \stackrel{L}{\leftrightarrow} 1$

Matlab

Verify Q9.9 in Matlab via

```
>> syms s
>> y = ilaplace(2*(s+2)/(s^2+7*s+12))
>> t = 0:0.05:2;
>> y1 = subs(y);
>> plot(t,y1);
```

Do the same for the other examples in the help section for ilaplace. Note that in Matlab `dirac(t)` is the impulse/delta function $\delta(t)$ and `heaviside(t)` is the step function $u(t)$