EE-232 Signals & Systems

Lecture 14
Laplace Transform Properties

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Lecture 14: Laplace Transform Properties

Laplace transform (3 lectures):

Laplace transform as Fourier transform with convergence factor. **Properties of the Laplace transform**

Specific objectives for today:

- Linearity and time shift properties
- Convolution property
- Time domain differentiation & integration property
- Transforms table

Reminder: Laplace Transforms

- Equivalent to the Fourier transform when $s{=}j\omega$

$$X(s)=\int_{-\infty}^{\infty}x(t)e^{-st}dt$$ *Laplace transform*

$$x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}ds$$ *Inverse Laplace transform*

$$x(t)\overset{L}{\leftrightarrow}X(s)$$

- Associated region of convergence for which the integral is finite
- Used to understand the frequency characteristics of a signal (system)
- Used to solve ODEs because of their convenient calculus and convolution properties (today)

Linearity of the Laplace Transform

If $x_1(t) \overset{L}{\leftrightarrow} X_1(s)$ ROC=R_1

and $x_2(t) \overset{L}{\leftrightarrow} X_2(s)$ ROC=R_2

Then $ax_1(t) + bx_2(t) \overset{L}{\leftrightarrow} aX_1(s) + bX_2(s)$ ROC= $R_1 \cap R_2$

This follows directly from the definition of the Laplace transform (as the integral operator is linear). It is easily extended to a linear combination of an arbitrary number of signals

Time Shifting & Laplace Transforms

If $x(t) \overset{L}{\leftrightarrow} X(s)$ ROC=R

Then $x(t-t_0) \overset{L}{\leftrightarrow} e^{-st_0} X(s)$ ROC=R

Proof $x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$

Now replacing t by $t\text{-}t_0$

$$x(t-t_0) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{s(t-t_0)} ds$$

$$= \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} (e^{-st_0} X(s)) e^{st} ds$$

Recognising this as

$$L\{x(t-t_0)\} = e^{-st_0} X(s)$$

A signal which is shifted in time may have both the **magnitude and the phase** of the Laplace transform altered.

Example: Linear and Time Shift

Consider the signal (linear sum of two time shifted sinusoids)

$$x(t) = 2x_1(t-2.5) - 0.5x_1(t-4)$$

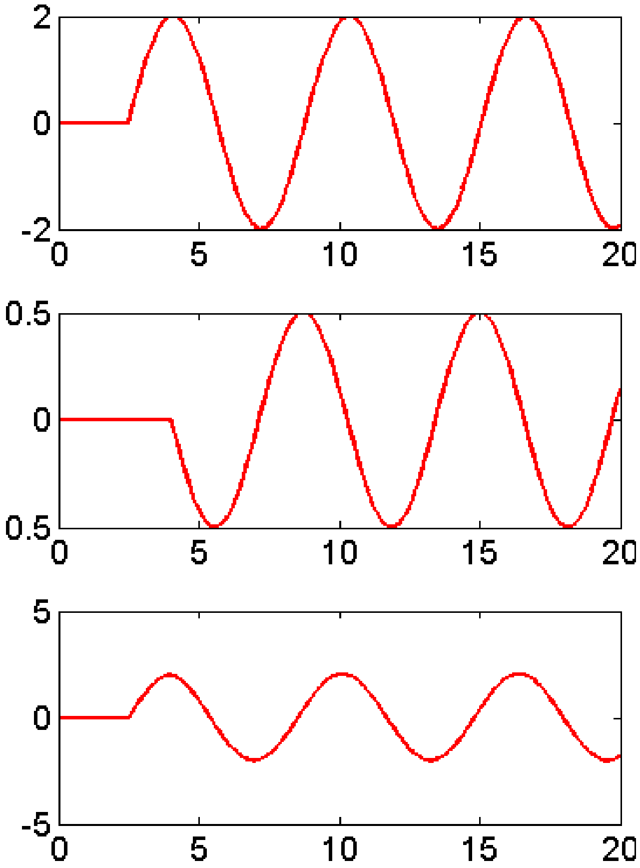
where $x_1(t) = \sin(\omega_0 t)u(t)$.

Using the sin() Laplace transform example

$$X_1(s) = \frac{\omega_0}{s^2+\omega_0^2} \qquad \text{Re}\{s\} > 0$$

Then using the **linearity** and **time shift** Laplace transform properties

$$X(s) = \left(2e^{-2.5s} - 0.5e^{-4s}\right)\frac{\omega_0}{s^2+\omega_0^2} \qquad \text{Re}\{s\} > 0$$



Convolution

The Laplace transform also has the multiplication property, i.e.

$$x(t) \overset{L}{\leftrightarrow} X(s) \qquad \text{ROC}=R_1$$

$$h(t) \overset{L}{\leftrightarrow} H(s) \qquad \text{ROC}=R_2$$

$$x(t) * h(t) \overset{L}{\leftrightarrow} X(s)H(s) \qquad \text{ROC} \supseteq R_1 \cap R_2$$

Proof is "identical" to the Fourier transform convolution

Note that pole-zero cancellation may occur between *H*(*s*) and *X*(*s*) which extends the ROC

$$X(s) = \frac{s+1}{s+2} \qquad \Re\{s\} > -2$$

$$H(s) = \frac{s+2}{s+1} \qquad \Re\{s\} > -1$$

$$X(s)H(s) = 1 \qquad -\infty < \Re\{s\} < \infty$$

Example 1: 1^{st} Order Input & First Order System Impulse Response

Consider the Laplace transform of the output of a first order system when the input is an exponential (decay?)

$$x(t) = e^{-at}u(t)$$

$$h(t) = e^{-bt}u(t)$$

Solved with Fourier transforms when a,b>0

Taking Laplace transforms

$$X(s) = \frac{1}{s+a} \qquad \mathrm{Re}\{s\} > -a$$

$$H(s) = \frac{1}{s+b}, \quad \mathrm{Re}\{s\} > -b$$

Laplace transform of the output is

$$Y(s) = \frac{1}{s+a}\frac{1}{s+b} \qquad \mathrm{Re}\{s\} > \max\{-a,-b\}$$

Example 1: Continued …

Splitting into partial fractions

$$Y(s) = \left(\frac{1}{b-a}\right)\left(\frac{1}{s+a} - \frac{1}{s+b}\right) \qquad \mathrm{Re}\{s\} > \max\{-a, -b\}$$

and using the inverse Laplace transform

$$y(t) = \tfrac{1}{b-a}\left(e^{-at}u(t) - e^{-bt}u(t)\right)$$

Note that this is the same as was obtained earlier, expect it is **valid for all** *a* & *b*, i.e. we can use the Laplace transforms to solve ODEs of LTI systems, using the system's impulse response

$$h(t) \overset{L}{\leftrightarrow} H(s)$$

Example 2: Sinusoidal Input

Consider the 1st order (possible unstable) system response with input $x(t)$

$$h(t) = e^{-at}u(t)$$

$$x(t) = \cos(\omega_0 t)u(t)$$

Taking Laplace transforms

$$H(s) = \frac{1}{s+a} \qquad \text{Re}\{s\} > -a$$

$$X(s) = \frac{s}{s^2+\omega_0^2} \qquad \text{Re}\{s\} > 0$$

The Laplace transform of the output of the system is therefore

$$Y(s) = \frac{s}{s^2+\omega_0^2}\frac{1}{s+a} \qquad \text{Re}\{s\} > \max\{0,-a\}$$

$$= \left(\frac{1}{a^2+\omega_0^2}\right)\frac{as+\omega_0^2}{s^2+\omega_0^2} + \left(\frac{-a}{a^2+\omega_0^2}\right)\frac{1}{s+a}$$

and the inverse Laplace transform is

$$y(t) = \frac{u(t)}{a^2+\omega_0^2}\left(a\sin(\omega_0 t) + \omega_0\cos(\omega_0 t) - ae^{-at}\right)$$

Differentiation in the Time Domain

Consider the Laplace transform derivative in the time domain

$$x(t) \overset{L}{\leftrightarrow} X(s) \qquad \text{ROC}=R$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}ds$$

$$\frac{dx(t)}{dt} \overset{L}{\leftrightarrow} sX(s) \qquad \text{ROC}\supseteq R$$

sX(*s*) has an extra zero at 0, and may cancel out a corresponding pole of *X*(*s*), so ROC may be larger

Widely used to solve when the system is described by LTI differential equations

Example: System Impulse Response

Consider trying to find the system response (potentially unstable) for a second order system with an impulse input $x(t)=\delta(t)$, $y(t)=h(t)$

$$a\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+cy(t)=x(t)$$

Taking Laplace transforms of both sides and using the linearity property

$$aL\left\{\frac{d^2y(t)}{dt^2}\right\}+bL\left\{\frac{dy(t)}{dt}\right\}+cL\left\{y(t)\right\}=L\{\delta(t)\}$$

$$L\{y(t)\}(as^2+bs+c)=1$$

$$L\{y(t)\}=H(s)=\frac{1}{as^2+bs+c}=\frac{1}{a(s-r_1)(s-r_2)}=\frac{k_1}{(s-r_1)}+\frac{k_2}{(s-r_2)}$$

where r_1 and r_2 are distinct roots, and calculating the inverse transform

$$y(t)=k_1e^{r_1t}u(t)+k_2e^{r_2t}u(t)$$

The general solution to a second order system can be expressed as the sum of two complex (possibly real) exponentials

Lecture 14: Summary

Like the Fourier transform, the Laplace transform is **linear** and represents time shifts (t-T) by multiplying by e^{-sT}

Convolution

$$x(t)*h(t) \overset{L}{\leftrightarrow} X(s)H(s) \qquad \text{ROC} \supseteq R_1 \cap R_2$$

Convolution in the time domain is equivalent to multiplying the Laplace transforms

Laplace transform of the system's impulse response is very important $H(s) = \int h(t)e^{-st}dt$. Known as the **transfer function**.

Differentiation

$$\frac{dx(t)}{dt} \overset{L}{\leftrightarrow} sX(s) \qquad \text{ROC} \supseteq R$$

Very important for solving ordinary differential equations

Questions

Theory

SaS, O&W, Q9.29-9.32

Work through slide 12 for the first order system

$$a\frac{dy(t)}{dt} + by(t) = \delta(t)$$

Where the aim is to calculate the Laplace transform of the impulse response as well as the actual impulse response

Matlab

Implement the systems on slides 10 & 12 in Simulink and verify their responses by exact calculation.

Note that `roots()` is a Matlab function that will calculate the roots of a polynomial expression