EE-232 Signals & Systems

Lecture 15

Continuous-Time Transfer Functions

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Continuous-Time Transfer Functions

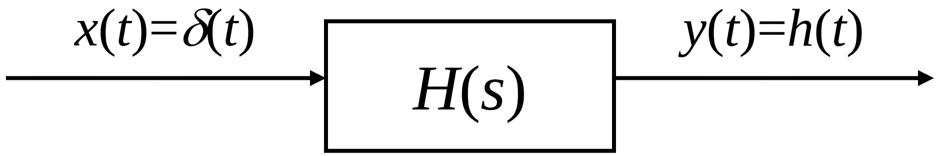
Transfer Function of Continuous-Time Systems (3 lectures): **Transfer function, frequency response,** Bode diagram. Physical realisation, stability. Poles and zeros, rubber sheet analogy.

Specific objectives for today:

- Transfer function of a system and examples
- Transient and steady-state behaviour
- Frequency response
- System gain/amplitude and phase margin

System/Transfer Functions

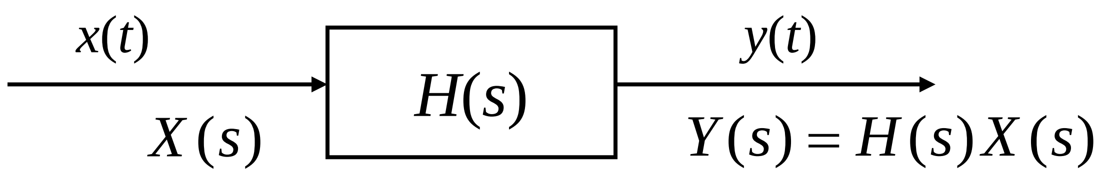
The system/transfer function, *H*(*s*), is defined as the Laplace transform of the system's impulse response



$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

When $s=j\omega$, this is the **Fourier transform** and more generally, this is the **Laplace transform**

The transfer function is very important because



the unknown system output (Laplace transform) is given by the **multiplication** *X*(*s*) and *H*(*s*)

Example 1: First Order System

Consider a general LTI, first order, differential equation with an impulse input

$$a\frac{dy(t)}{dt} + by(t) = x(t)$$

$$a\frac{dh(t)}{dt} + bh(t) = \delta(t)$$

Taking Laplace transforms

$$L\{h(t)\}(as+b) = 1$$

$$H(s) = L\{h(t)\} = \frac{1}{a(s+b/a)} \qquad \mathrm{Re}\{s\} > -b/a$$

which gives the system's transfer function, *H*(*s*). This can be solved to show that (see earlier examples)

$$h(t) = a^{-1}e^{-(b/a)t}u(t)$$

Example 2: Second Order System

Consider a general LTI, second order, differential equation with an impulse input

$$a\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+cy(t)=x(t)$$

$$a\frac{d^2h(t)}{dt^2}+b\frac{dh(t)}{dt}+ch(t)=\delta(t)$$

Taking Laplace transforms

$$L\{h(t)\}(as^2+bs+c)=1 \qquad \mathrm{Re}\{s\}>\max\{\mathrm{Re}\{-r_1\},\mathrm{Re}\{-r_2\}\}$$

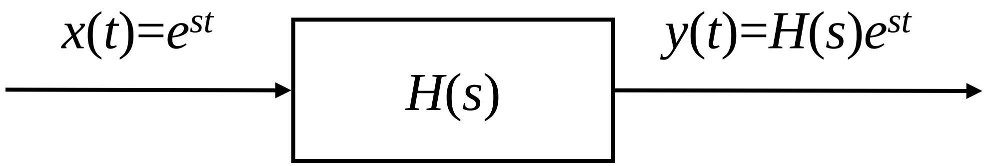
$$H(s)=L\{h(t)\}=\frac{1}{as^2+bs+c}=\frac{1}{a(s-r_1)(s-r_2)}$$

which gives the **system's transfer function**, *H*(*s*). This can be solved, using partial fractions, to show that

$$h(t)=a^{-1}k_1e^{r_1t}u(t)+a^{-1}k_2e^{r_2t}u(t)$$

Transfer Functions & System Eigenfuctions

Remember that $x(t)=e^{st}$ is an eigenfuction of an LTI system with corresponding eigenvalue $H(s)$



In addition, by Laplace theory, most input signals $x(t)$ can be expressed as a linear combination of basis signals e^{st} (inverse Laplace transform):

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

Therefore the system's output can be expressed as

$$y(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} H(s)X(s)e^{st}ds = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} Y(s)e^{st}ds$$

which is why $Y(s)=H(s)X(s)$, again using the transfer function

Frequency Response Analysis

Of particular interest is **frequency response analysis**. This corresponds to input signals of the form

$$x(t) = e^{j\omega t}$$

and the corresponding transfer function is

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

In this case $s = j\omega$, the **transfer function** corresponds to the **Fourier transform**. It is a complex function of frequency.

Example when *x*(*t*) is periodic, with fundamental frequency ω_0, the Fourier transform is given by:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

The system's response is given by

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

Example: 1st Order System and cos() Input

The transfer function of a 1st order system is given by:

$$H(j\omega) = \frac{1}{(j\omega + b)}$$

Assume a=1

The input signal $x(t)=\cos(\omega_0 t)$, which has fundamental frequency ω_0 is:

$$x(t) = \tfrac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

The (stable) system's output is:

$$y(t) = \tfrac{1}{2}\left(e^{j\omega_0 t}\frac{1}{b + j\omega_0} + e^{-j\omega_0 t}\frac{1}{b - j\omega_0} \right)$$

$$= \tfrac{1}{2(b^2+\omega_0^2)}\left(e^{j\omega_0 t}(b - j\omega_0) + e^{-j\omega_0 t}(b + j\omega_0) \right)$$

$$= \tfrac{b}{(b^2+\omega_0^2)}\cos(\omega_0 t) + \tfrac{\omega_0}{(b^2+\omega_0^2)}\sin(\omega_0 t)$$

Gain and Phase Transfer Function Analysis

For a complex number/function, we can represent it in polar form by calculating the magnitude (gain) and angle (phase):

$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$$

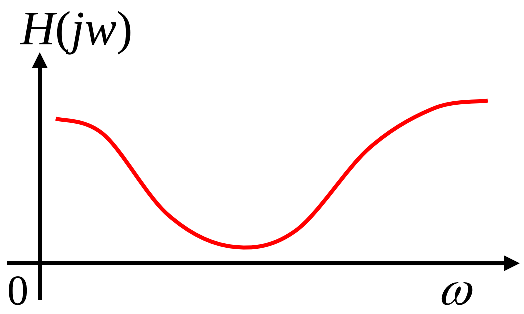
In filter/system analysis and design, we're interested in how certain frequencies are magnified or suppressed -

These are of particular interest by plotting the system properties (amplitude/phase) against frequency:

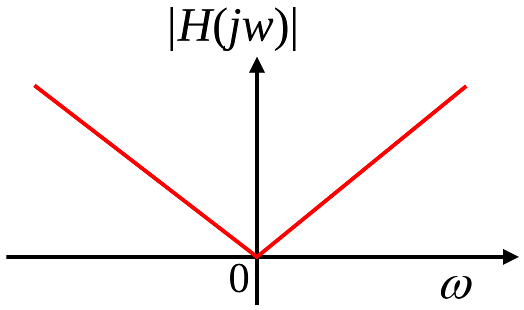
- Frequency shaping
- Frequency selection
- Low pass
- High pass

Frequency Shaping Filters

It is often necessary to change the relative magnitude of a signal at different frequencies, which is referred to as **filtering**. LTI systems that change the shape of the spectrum are referred to as **frequency shaping filters**.



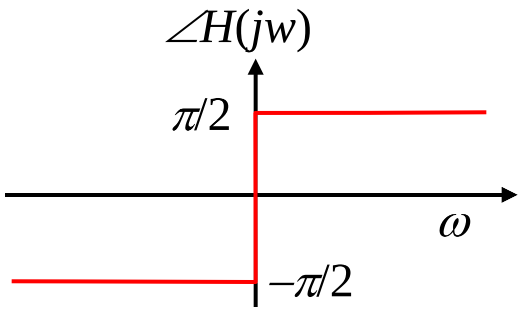
Audio systems often contain frequency shaping filters (LTI systems) to change the relative amount of bass (low frequency) and treble (high frequency). Real valued and often plotted on log scaling (dB = $20\log_{10}|H(j\omega)|$)



Complex differentiating filters are defined by

$$H(j\omega) = j\omega$$

which are useful for enhancing rapid variations in a signal. Both the magnitude and phase are plotted against frequency

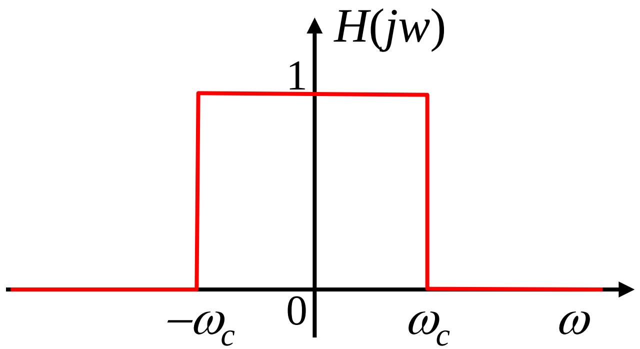


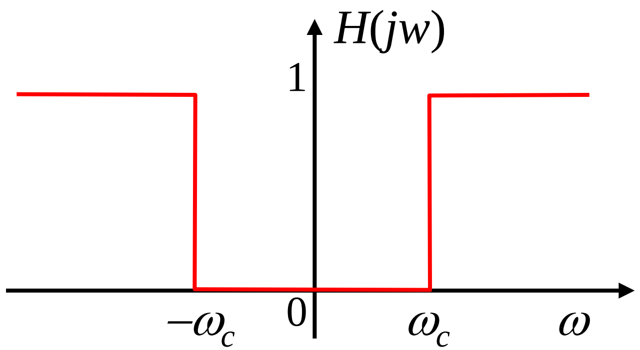
Frequency Selection Filters

Frequency selection filters are specially designed to accept some frequencies and reject others. Noise in an audio recording can be removed by low pass filtering, multiple communication signals can be encoded at different frequencies and then recovered by selecting particular frequencies

Low pass filters are designed to reject/attenuate high frequency "noise" while passing on the low frequencies

High pass filters are designed to reject/attenuate low frequency signal components while passing on high frequency

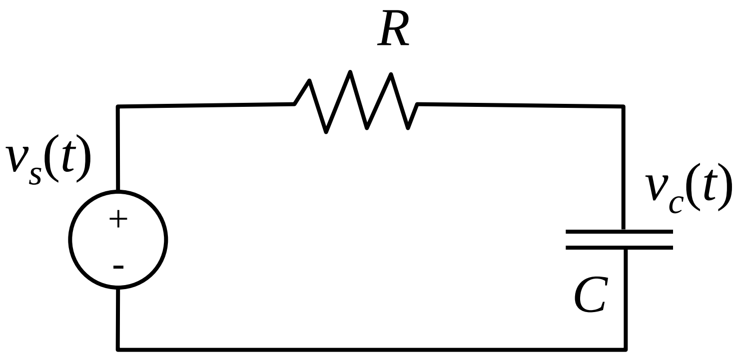




Electrical Low Pass Filter

Differential equation for the LTI system

$$RC\frac{dv_c(t)}{dt}+v_c(t)=v_s(t)$$

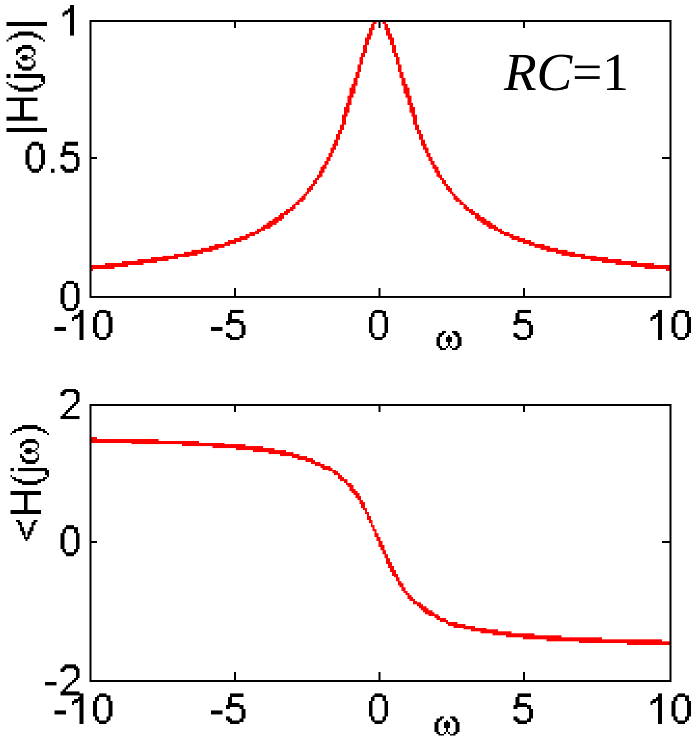


The frequency response transfer function $H(j\omega)$ can be determined using the eigensystem property **or** using its impulse response definition

$$RCj\omega H(j\omega)+H(j\omega)=1$$

$$H(j\omega)=\frac{1}{1+j\omega RC}$$

Magnitude-phase plot shown right



Step response:

$$v_c(t)=(1-e^{-t/RC})u(t)$$

High RC – good frequency selection

Low RC – fast time response

Inevitable time/frequency design compromise

Electrical High Pass Filter

Differential equation for the LTI system

$$RC\frac{dv_r(t)}{dt} + v_r(t) = RC\frac{dv_s(t)}{dt}$$

The frequency response transfer function *H*(*jω*) can be determined using eigenfunction property or impulse response

$$RCj\omega H(j\omega) + H(j\omega) = RCj\omega.1$$

$$H(j\omega) = \frac{j\omega RC}{1 + j\omega RC}$$

Magnitude-phase plot shown right

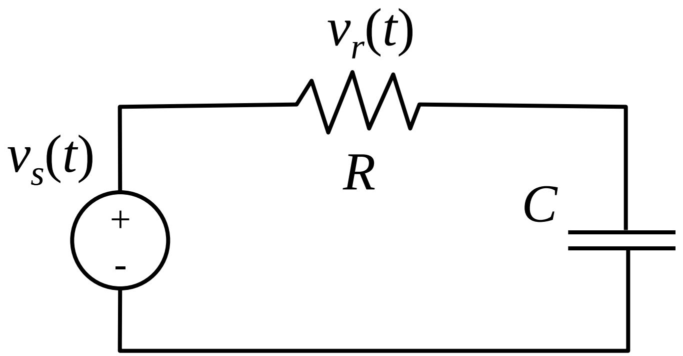
Step response:

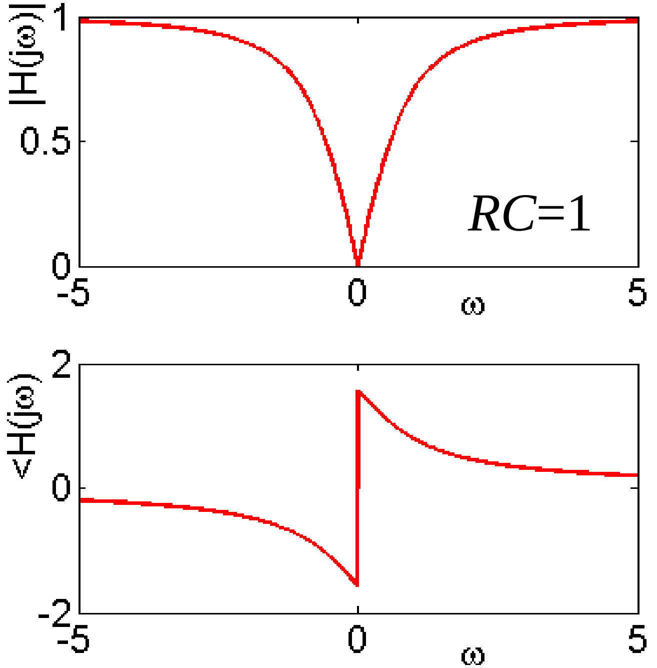
$$v_c(t) = e^{-t/RC}u(t)$$

High RC – good frequency selection

Low RC – fast time response

Inevitable time/frequency design compromise





Summary

The system's **transfer function $H(s)$** uniquely determines an LTI because, using convolution, it is possible to determine the output Fourier/Laplace transform, $Y(s)$, given the input Fourier/Laplace transform $X(s)$

When the system is a LTI ODE, the transfer function is a **rational function of s**, and it can be inverted (solving the ODE) by expressing the transfer function as **partial fractions**

In this case, the impulse response of an ODEs is a **sum of complex exponentials**

Frequency response analysis, by investigating the **magnitude and phase** at different frequencies, is a standard method for system design

ODEs can be used to implement **approximate high pass and low pass filters**