EE-232 Signals & Systems

Lecture 16

Continuous-Time Transfer Functions

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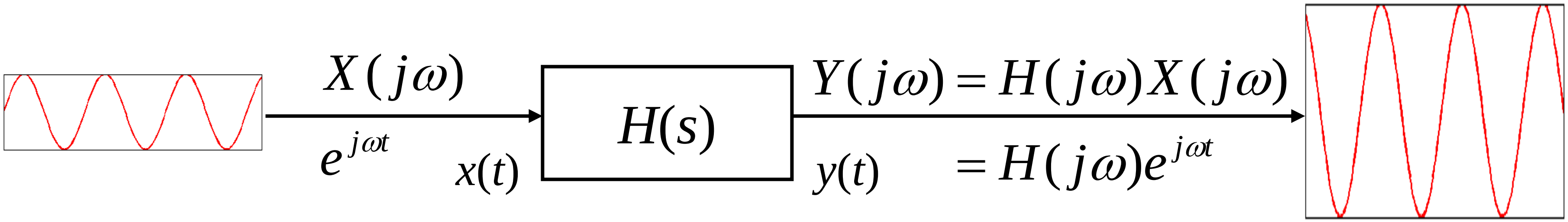
Continuous-Time Transfer Functions

Transfer Function of Continuous-Time Systems (3 lectures): Transfer function, **frequency response, Bode diagram.** Physical realisation, stability. Poles and zeros, rubber sheet analogy.

Specific objectives for today:

- Transfer functions and **frequency response**
- **Bode diagrams**

Introduction: Transfer Functions & Frequency Response



We can use the Fourier (Laplace) transfer function $H(j\omega)$ ($H(s)$) in a variety of ways:

- Design a system/filter with appropriate frequency domain characteristics
- Calculate the system’s time domain response using $Y(j\omega)=H(j\omega)X(j\omega)$ and taking the inverse Fourier transform

However, we can also get a lot of information from studying $H(j\omega)$ directly and representing it in polar fashion as

$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$$

Example: 1st Order System and Cos Input

The 1st order system transfer function is: (a>0, $h(t)=e^{-at}u(t)$)

$$H(j\omega)=\frac{1}{(j\omega+a)}$$

The input signal $x(t)=\cos(\omega_0 t)$, which has fundamental frequency ω_0, has Fourier transform:

$$X(j\omega)=\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$$

The (stable) system's output is:

$$Y(j\omega)=\frac{\pi}{a+j\omega}\delta(\omega-\omega_0)+\frac{\pi}{a+j\omega}\delta(\omega+\omega_0)$$

$$=\frac{\pi(a-j\omega)}{a^2+\omega^2}\delta(\omega-\omega_0)+\frac{\pi(a-j\omega)}{a^2+\omega^2}\delta(\omega+\omega_0)$$

$$y(t)=\tfrac{1}{2\pi}\int_{-\infty}^{\infty}Y(j\omega)e^{j\omega t}d\omega=\frac{(a-j\omega_0)e^{j\omega_0 t}+(a+j\omega_0)e^{-j\omega_0 t}}{2(a^2+\omega_0^2)}$$

$$=\frac{a\cos(\omega_0 t)+\omega_0\sin(\omega_0 t)}{(a^2+\omega_0^2)}$$

System Transient & Steady State Response

Compare with the example from lecture 14

$$h(t) = e^{-at}u(t)$$

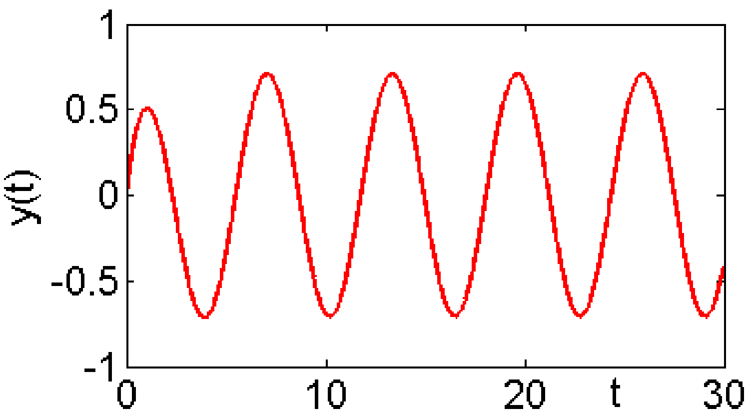
$$x(t) = \cos(\omega_0 t)u(t)$$

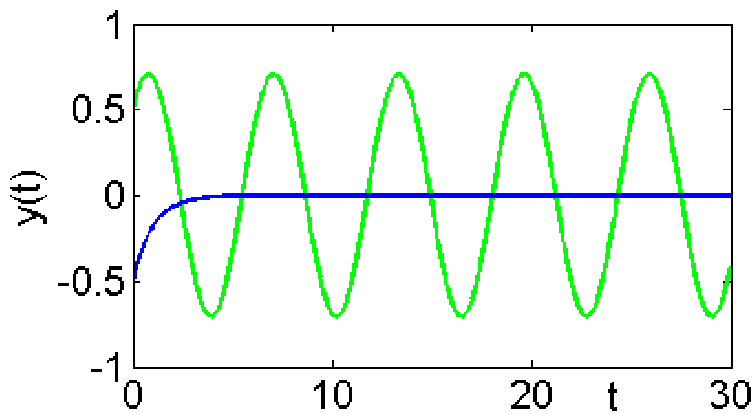
which was solved using the Laplace transform

$$y(t) = \frac{u(t)}{a^2 + \omega_0^2}\left(a\cos(\omega_0 t) + \omega_0 \sin(\omega_0 t) - ae^{-at}\right)$$

This is composed of two parts:

Transient (blue) and **steady state/natural** (green - Fourier) responses





System Gain and Phase Shift

In the frequency domain, the effect of the system on the input signal for the frequency component ω is:

$$Y(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}\,|X(j\omega)|e^{j\angle X(j\omega)}$$

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)|$$

$$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

The effect of a system, $H(j\omega)$, has on the Fourier transform of an input signal is to:

- **Scale the magnitude** by $|H(j\omega)|$. This is commonly referred to as the **system gain**.
- **Shift the phase** of the input signal by adding $\angle H(j\omega)$ to it. This is commonly referred to as the **phase shift**.

These modifications (**magnitude and phase distortions**) may be desirable/undesirable and must be understood in system analysis and design.

Example: Cos Input to a 1st Order System

Consider a sinusoidal input signal to a first order, LTI, stable system

$$h(t) = e^{-at}u(t), \qquad a > 0$$

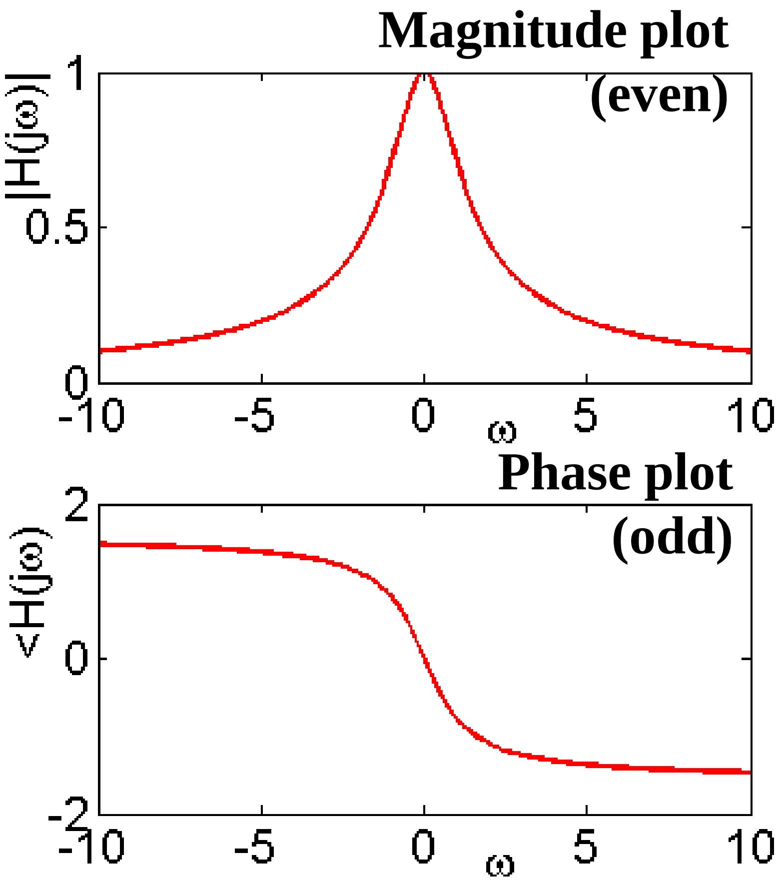
$$x(t) = \cos(\omega_0 t)$$

$$H(j\omega) = \frac{1}{j\omega + a}$$

When ω_0 is close to zero, its magnitude is passed on scaled by 1/*a*

When the $|\omega_0|$ is high, the signal is substantially suppressed

i.e. it is a low pass filter …



We deduce the properties solely by looking at the **transfer function in the frequency domain**

The Effect of Phase …

The effect of the transfer function's magnitude is fairly easy to see – it magnifies/suppresses the input signal

The effect of the change in phase is a bit less obvious to imagine.

Consider when the phase shift is a linear function of ω:

$$H(j\omega) = e^{-j\omega t_0} \qquad |H(j\omega)| = 1$$

$$\angle H(j\omega) = -\omega t_0$$

This system corresponds to a **pure time shift** of the input (see lectures 7,9,14)

$$y(t) = x(t-t_0)$$

Slope of the phase corresponds to the time delay

When the phase is not a linear function, it is slightly more complex

Log-Magnitude and Phase Plots

When analysing system responses, it is typical to use a **log scaling** for the magnitude

$$\log(|Y(j\omega)|) = \log(|H(j\omega)|) + \log(|X(j\omega)|)$$

So the gain effect is **additive**: **0 means “no change”**

If the log magnitude is plotted, the effect can be interpreted as adding each individual component (like the time-delayed phase)

Often units are **decibels (dB) 20log$_{10}$**

Similarly, taking logs of frequency allows us to view detail over a much greater range (which is important for frequency selective filters)

Note that taking a log of the frequency, we typically only consider positive frequency values (as the **magnitude is even**, and the **phase is odd**)

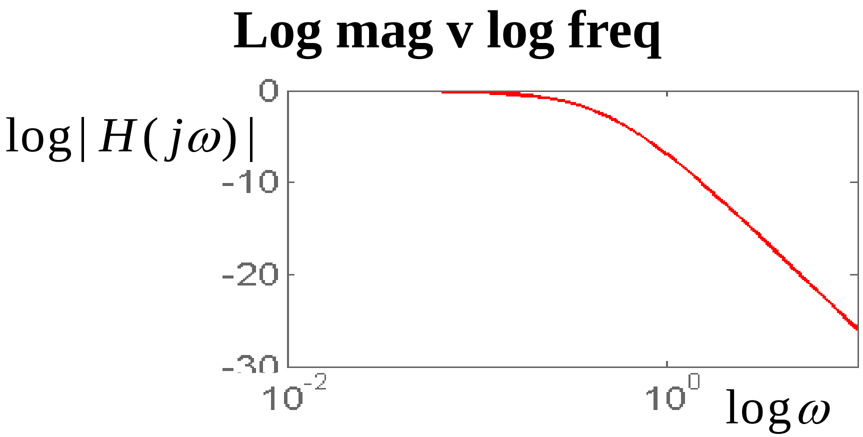
Bode Plots

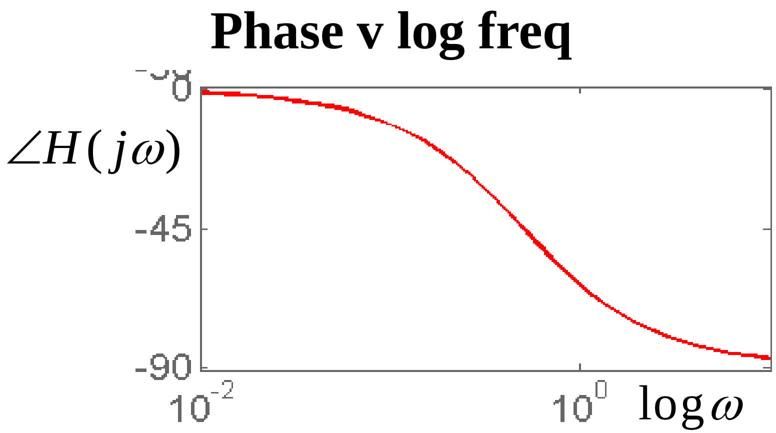
A **Bode Plot** for a system is simply plots of log magnitude and phase against log frequency

Both the log magnitude and phase effects are now **additive**

Widely used for **analysis and design** of **filters** and **controllers**

Example





Low pass, unity filter

Example 1: Bode Plot 1st Order System

Consider a LTI first order system described by:

$$\tau\frac{dy(t)}{dt}+y(t)=x(t), \qquad \tau>0$$

Fourier transfer function is:

$$H(j\omega)=\frac{1}{\tau j\omega+1}$$

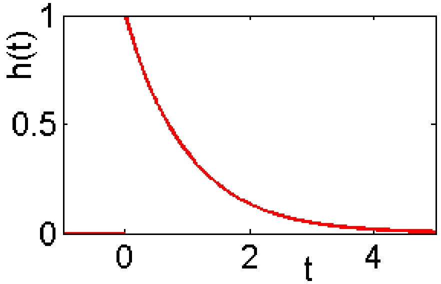
the impulse response is:

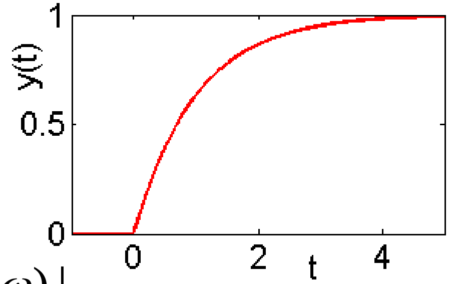
$$h(t)=\tfrac{1}{\tau}e^{-t/\tau}u(t)$$

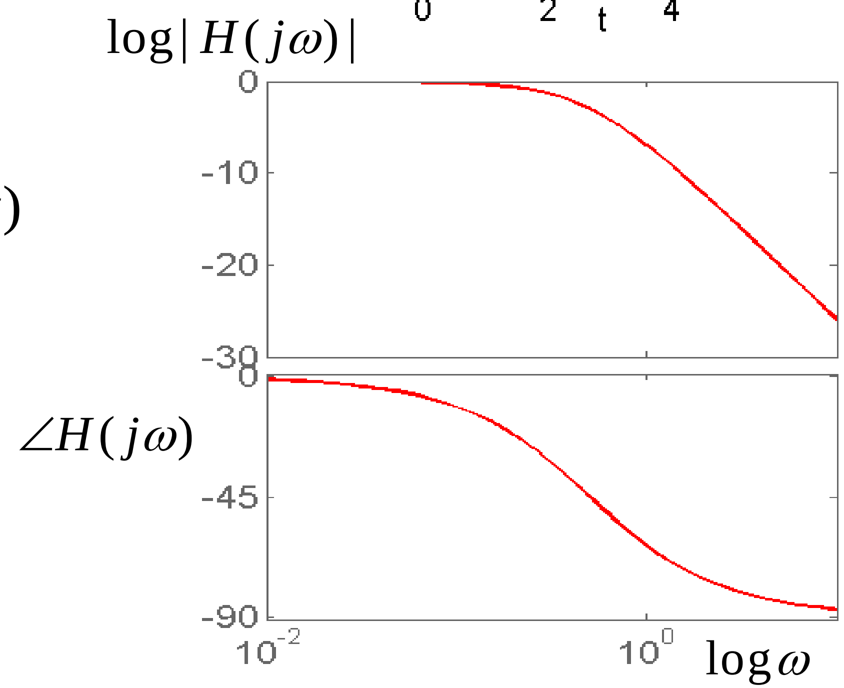
and the step response is:

$$y(t)=h(t)*u(t)=(1-\tfrac{1}{\tau}e^{-t/\tau})u(t)$$

Bode diagrams are shown as log/log plots on the x and y axis with τ=2.







Example 2: Bode Plot 2nd Order System

The LTI 2nd order differential equation

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

which can represent the response of mass-spring systems and RLC circuits, amongst other things

ω_n is the undamped natural frequency

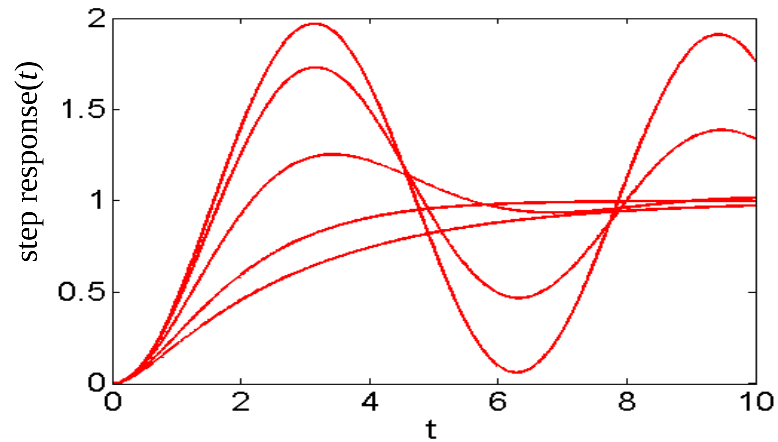
ζ is the damping ratio

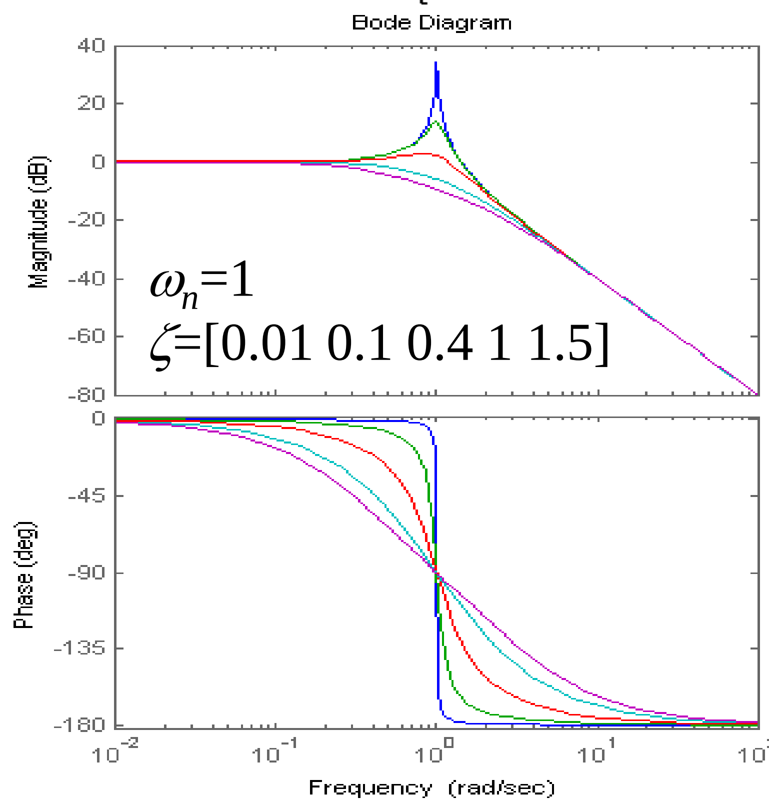
$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{\omega_n^2}{(j\omega - p_1)(j\omega - p_2)}$$

$$p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$p_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$





Lecture 16: Summary

A **frequency domain analysis** of the transfer function/Fourier transform is an important design/analysis concept

It can be understood in terms of

- $|H(j\omega)|$ - **magnitude** of the Fourier transform of the impulse response (transfer function)
- $\angle H(j\omega)$ – **phase** of the Fourier transform of the impulse response (transfer function)

Bode plots are plots of **log magnitude** and **phase** against **log frequency**.

- Used to plot a greater range of frequencies
- Used to plot decibel-type information
- Transfer function is now "additive"

Exercises

Theory

Verify the magnitude and phase plots on slide 7 by evaluating the 1st order transfer function for specific values of ω (=0, 1, 3, 5, 10), for *a*=1&10.

SaS, O&W, Q6.15, 6.18, 6.19, 6.27 & 6.28 (use Matlab for "sketching")

Matlab

1. Use Simulink to verify the transient/steady state response of a first order system described on Slide 5.

2. To perform a Bode plot of a first order system (slide 11),

Where τ=2

```
>> fbode([1], [2 1]);
```

Type `help fbode` to find out about the general structure.

Try doing a Bode plot for different values of the decay constant, say 1 and 100, what are the differences?

To perform a Bode plot of the second order system (slide 12)

```
>> fbode([1], [1 2 1]);
```

Again, try different values for the differential equation coeffs.