EE-232 Signals & Systems

Lecture 17

Continuous-Time Transfer Functions

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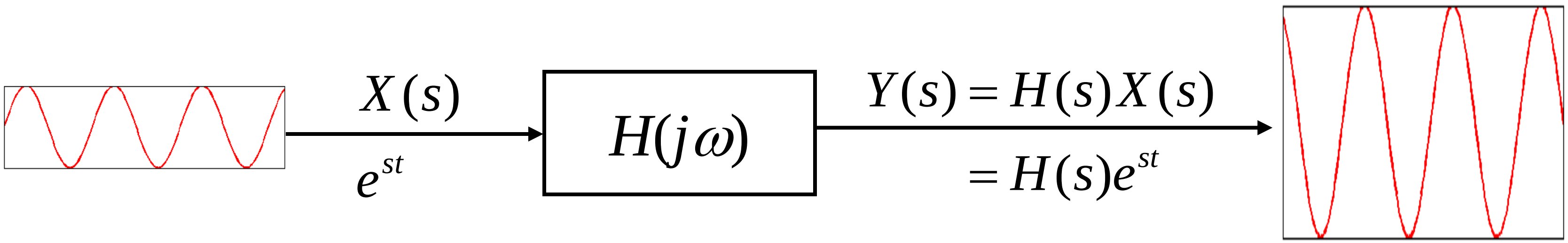
Continuous-Time Transfer Functions

Transfer Function of Continuous-Time Systems (3 lectures): Transfer function, frequency response, Bode diagram**. Physical realizability, stability. Poles and zeros, rubber sheet analogy.**

Specific objectives for today:

- System **causality** & transfer functions
- System **stability** & transfer functions
- Structures of sub-systems – series and feedback

Review: Transfer Functions, Frequency Response & Poles and Zeros



The system's transfer function is the Laplace (Fourier) transform of the system's impulse response $H(s)$ ($H(j\omega)$).

The transfer function's poles and zeros are $H(s) \propto \Pi_i(s-z_i)/\Pi_j(s-p_i)$.

This enables us to both calculate (from the differential equations) and analyse a system's response

Frequency response magnitude/phase decomposition

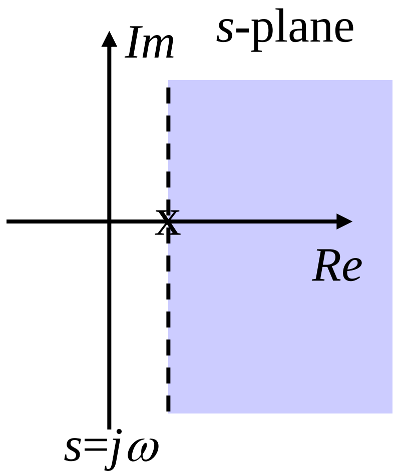
$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$$

Bode diagrams are a log/log plot of this information

System Causality & Transfer Functions

Remember, a system is causal if $y(t)$ only depends on $x(t)$, $dx(t)/dt$,…,$x(t-T)$ where $T>0$

This is equivalent to saying that an LTI system's impulse is $h(t) = 0$ whenever $t<0$.



Theorem The ROC associated with the (Laplace) transfer function of a causal system is a right-half plane

Note the converse is not necessarily true (but **is true** for a rational transfer function)

Proof By definition, for a causal system, $\sigma_0 \in$ROC:

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt = \int_{0}^{\infty} h(t)e^{-st}dt \quad \& \quad \int_{0}^{\infty} |h(t)|e^{-\sigma_0 t}dt < \infty$$

If this converges for σ_0, then consider any $\sigma_1 > \sigma_0$

$$\int_{0}^{\infty} |h(t)|e^{-\sigma_1 t}dt = \int_{0}^{\infty} |h(t)|e^{-\sigma_0 t}e^{-(\sigma_1+\sigma_0)t}dt \le \int_{0}^{\infty} |h(t)|e^{-\sigma_0 t}dt < \infty$$

so $\sigma_1 \in$ROC

Examples: System Causality

Consider the (LTI 1st order) system with an impulse response

$$h(t) = e^{-t}u(t)$$

This has a transfer function (Laplace transform) and ROC

$$H(s) = \frac{1}{s+1}, \quad \mathrm{Re}\{s\} > -1$$

The transfer function is rational and the ROC is a right half plane.
The corresponding **system is causal**.

Consider the system with an impulse response

$$h(t) = e^{-|t|}$$

The system transfer function and ROC

$$H(s) = \int_{-\infty}^{\infty} e^{-|t|} e^{-st} dt = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-st} dt + \int_{-\infty}^{\infty} e^{t}u(-t)e^{-st} dt$$

$$= \frac{1}{s+1} - \frac{1}{s-1} = \frac{-2}{s^2-1}, \qquad -1 < \mathrm{Re}\{s\} < 1$$

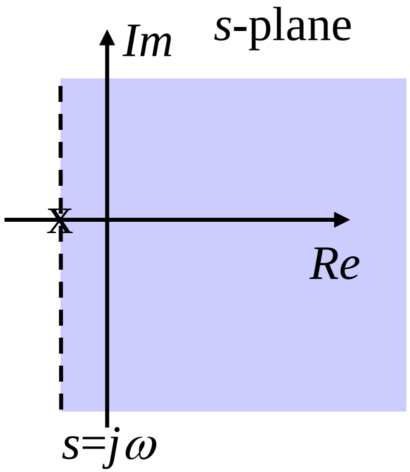
The ROC is **not** the right half plane, so the **system is not causal**

System Stability

Remember, a system is stable if $\forall x : |x| < U \rightarrow |y| < V$, which is equivalent to bounded input signal => bounded output

This is equivalent to saying that an LTI system's impulse is $\int |h(t)| dt < \infty$.

Theorem An LTI system is stable if and only if the ROC of $H(s)$ includes the entire $j\omega$ axis, i.e. $Re\{s\} = 0$.



Proof The transfer function ROC includes the "axis", $s=j\omega$ along which the Fourier transform has finite energy

Example The following transfer function is stable

$$e^{-at}u(t) \overset{L}{\leftrightarrow} X(s) = \frac{1}{s+a}, \quad \mathrm{Re}\{s\} > -a$$

Causal System Stability

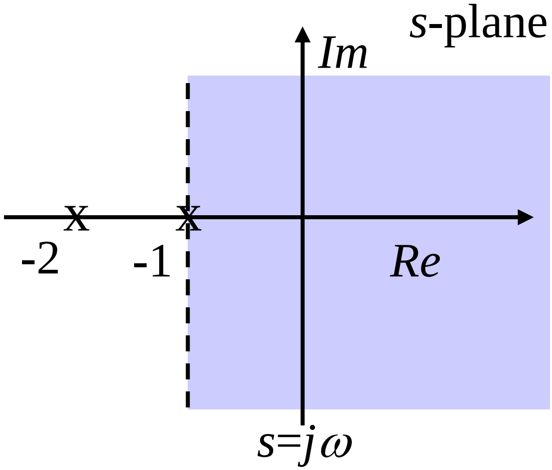
Theorem A causal system with rational system function $H(s)$ is stable if and only if all of the poles of $H(s)$ lie in the left-half plane of s, i.e. they have negative real parts

Proof Just combine the two previous theorems

Example

$$h(t) = (e^{-t} - e^{-2t})u(t)$$

$$H(s) = \frac{1}{(s+1)(s+2)}, \qquad \mathrm{Re}\{s\} > -1$$



Note that the poles of $H(s)$ correspond to the powers of the exponential response in the time domain. If the real part is negative, they exponential responses decay => stability. Also, the Fourier transform will exist and the imaginary axis lies in the ROC

LTI Differential Equation Systems

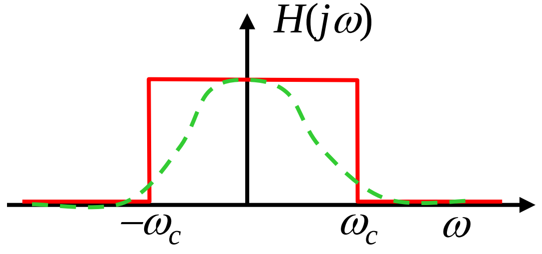
Physical and electrical systems are **causal**

Most physical and electrical systems dissipate energy, they are **stable**. The natural state is "at rest" unless some input/excitation signal is applied to the system

When performing analogue (continuous time) system design, the aim is to produce a time-domain "differential equation" which can then be translated to a known system (electrical circuit …)

This is often done in the frequency domain, which may/may not produce a causal, stable, time-domain differential equation.

Example: low pass filter



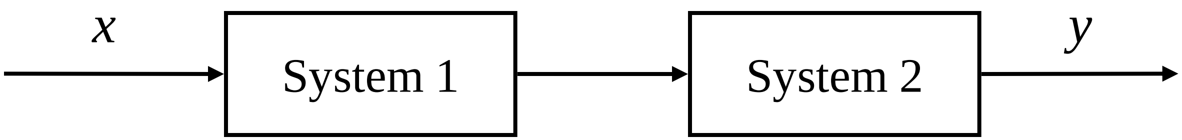
$$h(t) = \frac{\sin(\omega_c t)}{\pi t}$$

$$\frac{dh(t)}{dt} + ah(t) = \delta(t) \overset{F}{\leftrightarrow} \frac{1}{a + j\omega}$$

Structures of Sub-Systems

How to combine transfer functions $H_1(s)$ and $H_2(s)$ to get input output transfer function $Y(s) = H(s)X(s)$?

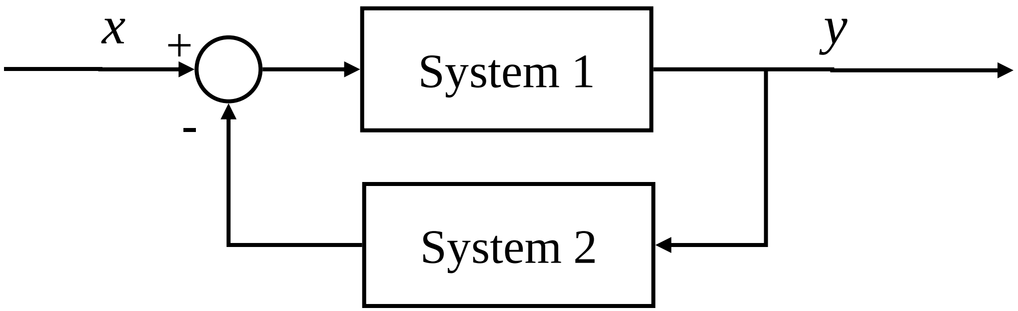
Series/cascade



$$H(s) = H_1(s)H_2(s)$$

Design $H_2()$ to cancel out the effects of $H_1()$

Feedback

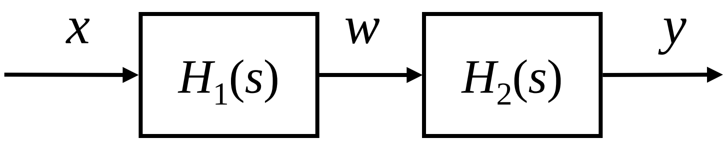


$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

Design $H_2()$ to regulate $y(t)$ to $x(t)$, so $H()=1$

Series Cascade & Feedback Proofs

Proof of Series Cascade transfer function

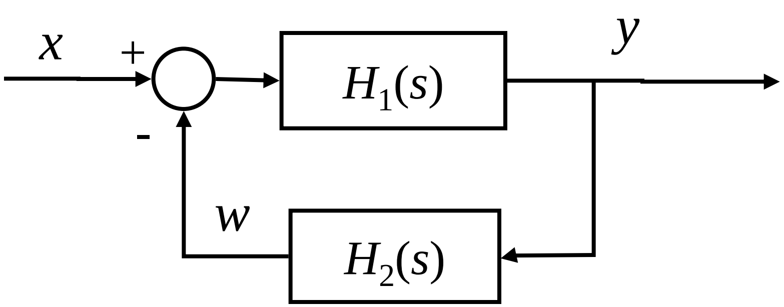


$$H(s) = H_1(s)H_2(s)$$

$$Y(s) = H_2(s)W(s), \qquad W(s) = H_1(s)X(s)$$

$$Y(s) = H_2(s)H_1(s)X(s)$$

Proof of Feedback transfer function



$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

$$W(s) = H_2(s)Y(s), \qquad Y(s) = H_1(s)(X(s) - W(s))$$

$$Y(s) = H_1(s)X(s) - H_1(s)H_2(s)Y(s)$$

$$Y(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}X(s)$$

Example: Cascaded 1st Order Systems

Consider two cascaded LTI first order systems

$$H_1(s) = \frac{1}{s+a}$$

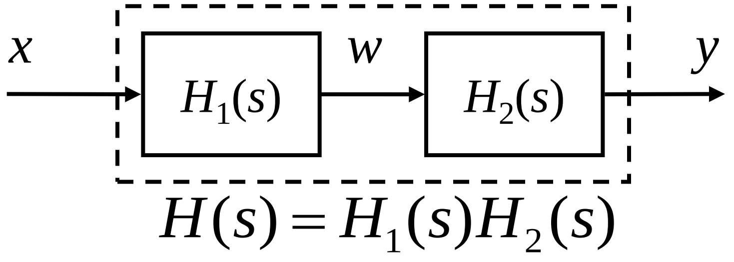
$$H_2(s) = \frac{1}{s+b}$$

$$H(s) = H_1(s)H_2(s)$$

$$= \frac{1}{s+a}\frac{1}{s+b}$$

$$= \frac{1}{s^2+(a+b)s+ab}$$

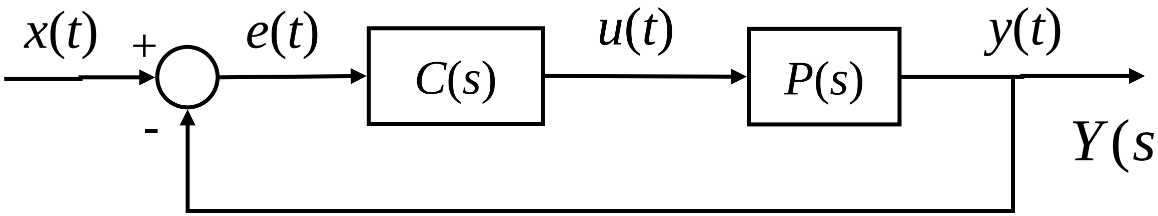
$$h(t) = \tfrac{1}{(b-a)}\left(e^{-at}-e^{-bt}\right)u(t)$$



The result of cascading two first order systems is a second order system. However, the roots of this quadratic are purely real (assuming *a* and *b* are real), so the output is not oscillatory, as would be the case with complex roots.

Example: Feedback Control

The idea of feedback is central for control (next semester)



$$Y(s) = \frac{P(s)C(s)}{1+P(s)C(s)} X(s)$$

The aim is to design the controller $C(s)$, such that the **closed loop** response, $Y(s)$, has particular characteristics

The plant $P(s)$ is the physical/electrical system (transfer function of differential equation) that must be controlled by the signal $u(t)$

The aim is to regulate the plant's response $y(t)$ so that it follows the demand signal $x(t)$

The error $e(t)=x(t)-y(t)$ gives an idea of the tracking performance

Real-world example

Control an aircraft's ailerons so that it follows a particular trajectory

Example Continued … High Gain Feedback

Simple control scheme (high gain feedback),

$C(s)=k>>0$

$u(t) = ke(t)$

For this controller, the system's response

$$Y(s) = \frac{kP(s)}{1+kP(s)} X(s)$$

$$\approx X(s)$$

as desired, when *k* is extremely large

The controller can be an operational amplifier

While this is a simple controller, it can have some disadvantages.

Lecture 17: Summary

System properties such as **stability**, **causality**, … can be interpreted in terms of the time domain (lecture 3), impulse response (lecture 6) or **transfer function** (this lecture).

For system **causality** the ROC must be a right-half plane

For system **stability**, the ROC must include the $j\omega$ axis

For **causal stability**, the ROC must include Re{s}>-ε

We can use the block transfer notation to calculate the transfer functions of serial, parallel and feedback systems.

Often the aim is to design a sub-system so that the overall transfer function has particular properties

Exercises

Theory

Prove the closed loop transfer function on Slide 12

SaS, O&W, 9.15, 9.16, 9.17, 9.18

Matlab

Verify the cascaded response on Slide 11 in Simulink, by cascading two first order models and comparing the response with the equivalent 2nd order model (i.e. pick values for *a* and *b* (which are not equal)),

NB the Continuous-System Simulink notation is of the form 1/s, s, 1/(s+a), I.e. the system blocks can be expressed as transfer functions and they can be chained together which just **multiplies** the individual transfer functions.