EE-232 Signals & Systems

Lecture 5

Linear Systems and Convolution

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Lecture 5: Linear Systems and Convolution

2. **Linear systems, Convolution** (3 lectures): **Impulse response, input signals as continuum of impulses. Convolution**, discrete-time and **continuous-time**. LTI systems and convolution

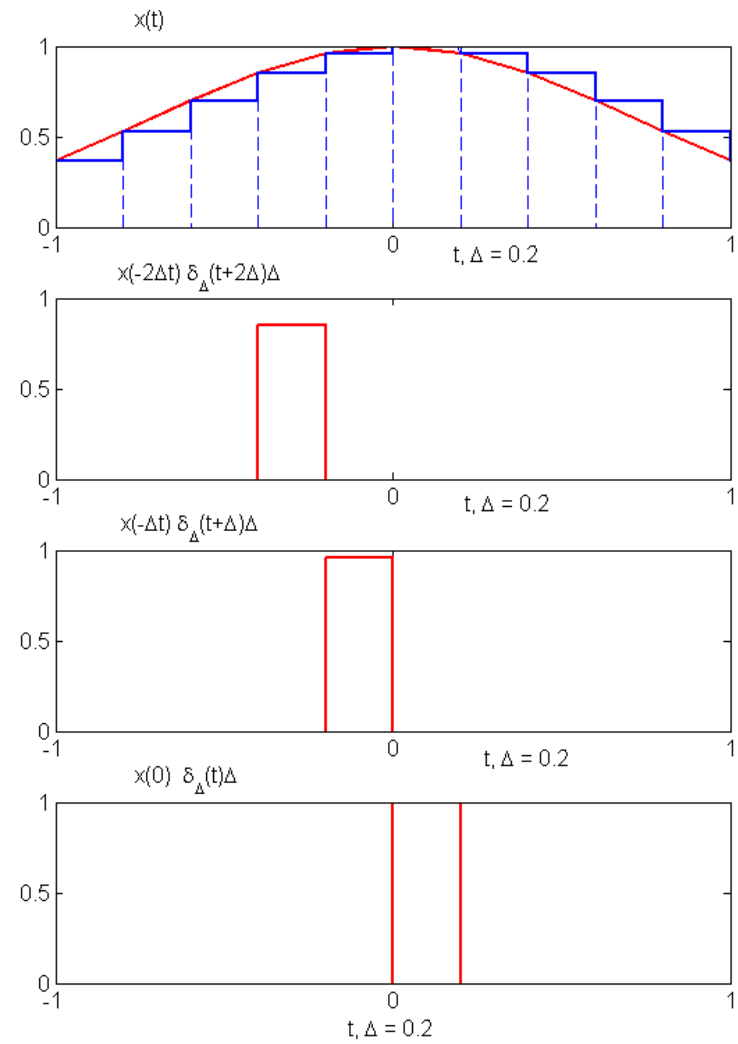
Specific objectives for today:

We're looking at **continuous time** signals and systems

- Understand a system's **impulse response** properties
- Show how any input signal can be decomposed into a **continuum of impulses**
- **Convolution**

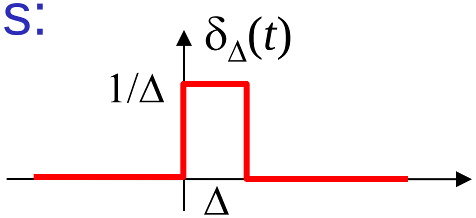
Introduction to "Continuous" Convolution

- In this lecture, we're going to understand how the convolution theory can be applied to continuous systems. This is probably most easily introduced by considering the relationship between discrete and continuous systems.
- The convolution sum for discrete systems was based on the **sifting** principle, the input signal can be represented as a superposition (linear combination) of scaled and shifted impulse functions.
- This can be generalised to continuous signals, by thinking of it as the limiting case of arbitrarily thin pulses



Signal "Staircase" Approximation

As previously shown, any continuous signal can be approximated by a linear combination of thin, delayed pulses:



$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & 0 \le t < \Delta \\ 0 & \text{otherwise} \end{cases}$$

Note that this pulse (rectangle) has a unit integral. Then we have:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

Only one pulse is non-zero for any value of *t*. Then as $\Delta \to 0$

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

When $\Delta \to 0$, the summation approaches an integral

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

This is known as the **sifting property** of the continuous-time impulse and there are an infinite number of such impulses $\delta(t-\tau)$

Alternative Derivation of Sifting Property

The unit impulse function, $\delta(t)$, could have been used to directly derive the sifting function.

$$\delta(t-\tau)=0 \quad t\neq\tau$$

$$\int_{-\infty}^{\infty}\delta(t-\tau)d\tau=1$$

Therefore:

$$x(\tau)\delta(t-\tau)=0 \quad t\neq\tau$$

$$\begin{aligned}\int_{-\infty}^{\infty}x(\tau)\delta(t-\tau)d\tau &= \int_{-\infty}^{\infty}x(t)\delta(t-\tau)d\tau \\ &= x(t)\int_{-\infty}^{\infty}\delta(t-\tau)d\tau \\ &= x(t)\end{aligned}$$

The previous derivation strongly emphasises the close relationship between the structure for **both** discrete and continuous-time signals

Linear Time Invariant Convolution

For a linear, time invariant system, all the impulse responses are simply time shifted versions:

$$h_\tau(t) = h(t-\tau)$$

Therefore, convolution for an LTI system is defined by:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

This is known as the **convolution integral** or the **superposition integral**

Algebraically, it can be written as:

$$y(t) = x(t) * h(t)$$

To evaluate the integral for a specific value of *t*, obtain the signal $h(t-\tau)$ and multiply it with $x(\tau)$ and the value $y(t)$ is obtained by integrating over τ from $-\infty$ to ∞.

Demonstrated in the following examples

Example 1: CT Convolution

Let $x(t)$ be the input to a LTI system with unit impulse response $h(t)$:

$$x(t) = e^{-at}u(t) \qquad a > 0$$

$$h(t) = u(t)$$

For $t>0$:

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau} & 0 < \tau < t \\ 0 & \text{otherwise} \end{cases}$$

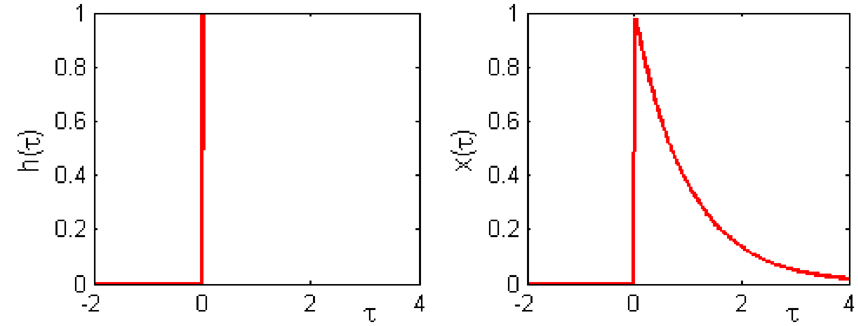
We can compute $y(t)$ for $t>0$:

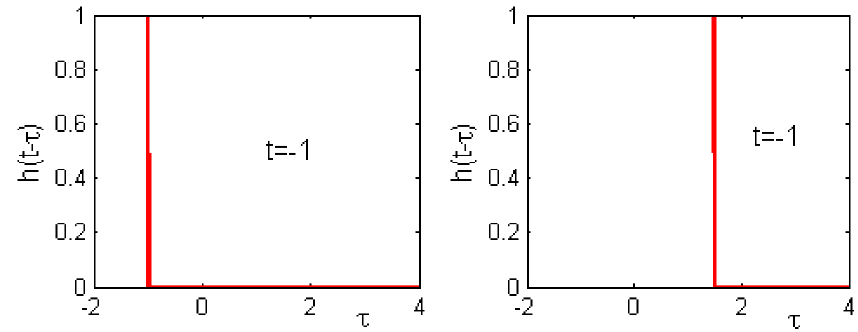
$$y(t) = \int_0^t e^{-a\tau}d\tau = -\tfrac{1}{a}e^{-a\tau}\Big|_0^t$$

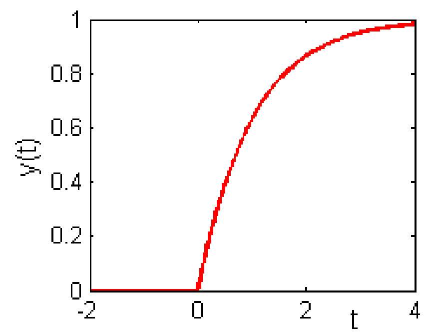
$$= \tfrac{1}{a}\left(1 - e^{-at}\right)$$

So for all t:

$$y(t) = \tfrac{1}{a}\left(1 - e^{-at}\right)u(t)$$







In this example a=1

Example 2: CT Convolution

Calculate the convolution of the following signals

$$x(t) = e^{2t}u(-t)$$

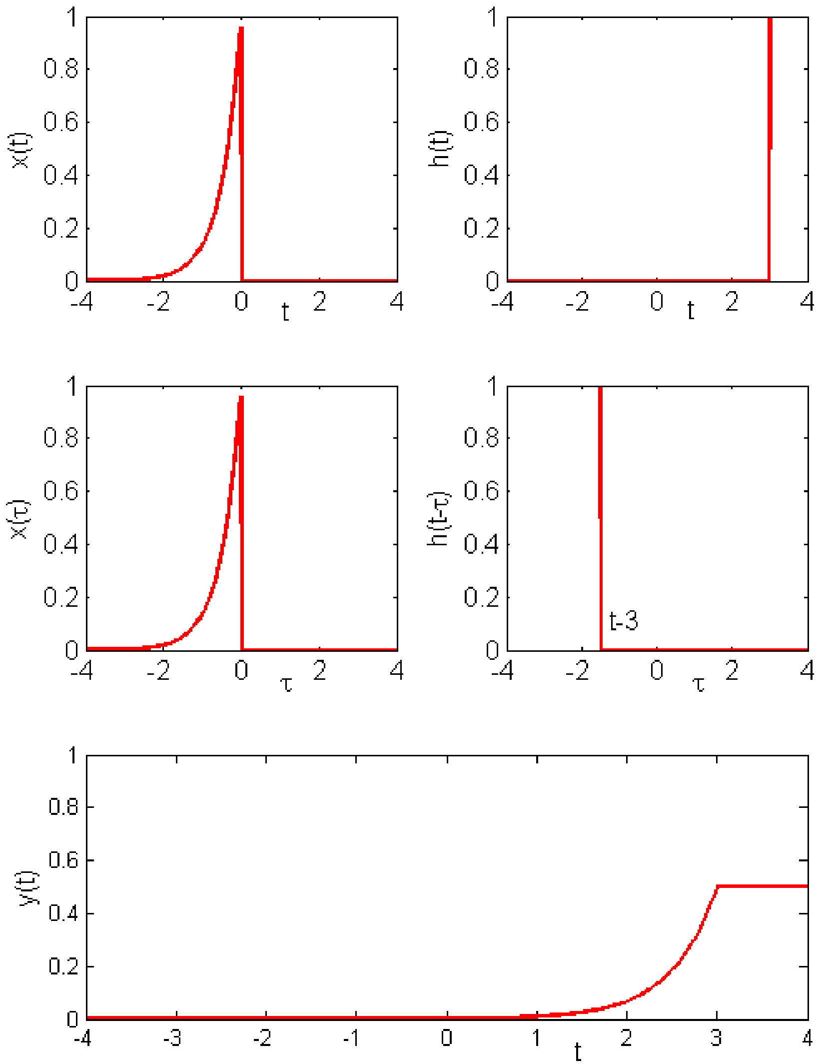
$$h(t) = u(t-3)$$

The convolution integral becomes:

$$y(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau = \tfrac{1}{2} e^{2(t-3)}$$

For t-3≥0, the product $x(\tau)h(t-\tau)$ is non-zero for -∞<τ<0, so the convolution integral becomes:

$$y(t) = \int_{-\infty}^{0} e^{2\tau} d\tau = \tfrac{1}{2}$$



Lecture 5: Summary

A continuous signal $x(t)$ can be represented via the sifting property:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Any continuous LTI system can be completely determined by measuring its unit impulse response $h(t)$

Given the input signal and the LTI system unit impulse response, the system's output can be determined via convolution via

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Note that this is an alternative way of calculating the solution $y(t)$ compared to an ODE. $h(t)$ contains the derivative information about the LHS of the ODE and the input signal represents the RHS.