EE-232 Signals & Systems

Lecture 6

Linear Systems and Convolution

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Lecture 6: Linear Systems and Convolution

Linear systems, Convolution: Impulse response, input signals as continuum of impulses. Convolution, discrete-time and continuous-time. LTI systems and convolution

Specific objectives for today:

- Properties of an LTI system
- Differential and difference systems

Rather than concentrate on the mechanics of how to calculate discrete/continuous-time convolution, we're looking at the impact on the system specification.

LTI Systems and Impulse Response

Any continuous/discrete-time LTI system is completely described by its impulse response through the convolution:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n]*h[n]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t)*h(t)$$

This only holds for LTI systems as follows:

Example: The discrete-time impulse response

$$h[n] = \begin{cases} 1 & n = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

Is completely described by the following LTI

$$y[n] = x[n] + x[n-1]$$

However, the following systems also have the same impulse response

$$y[n] = \left(x[n] + x[n-1]\right)^2$$

$$y[n] = \max\left(x[n], x[n-1]\right)$$

Therefore, if the system is non-linear, it is not completely characterised by the impulse response

Commutative Property

Convolution is a commutative operator (in both discrete and continuous time), i.e.:

$$x[n]*h[n] = h[n]*x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$x(t)*h(t) = h(t)*x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

For example, in discrete-time:

$$x[n]*h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{r=-\infty}^{\infty} x[n-r]h[r] = h[n]*x[n]$$

and similar for continuous time.

Therefore, when calculating the response of a system to an input signal $x[n]$, we can imagine the signal being convolved with the unit impulse response $h[n]$, or vice versa, whichever appears the most straightforward.

Distributive Property (Parallel Systems)

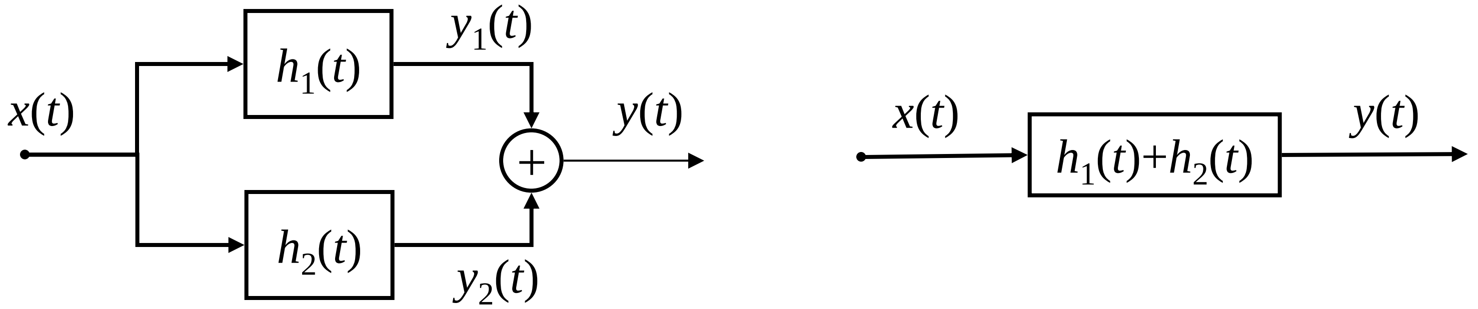
Another property of convolution is the distributive property

$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n] = y_1[n]+y_2[n]$$

$$x(t)*(h_1(t)+h_2(t)) = x(t)*h_1(t)+x(t)*h_2(t) = y_1(t)+y_2(t)$$

This can be easily verified

Therefore, the two systems:



are equivalent. The convolved sum of two impulse responses is equivalent to considering the two equivalent parallel system (equivalent for discrete-time systems)

Example: Distributive Property

Let $y[n]$ denote the convolution of the following two sequences:

$$x[n] = 0.5^n u[n] + 2^n u[-n]$$

$$h[n] = u[n]$$

$x[n]$ is non-zero for all n. We will use the **distributive** property to express $y[n]$ as the sum of two simpler convolution problems. Let $x_1[n] = 0.5^n u[n]$, $x_2[n] = 2^n u[-n]$, it follows that

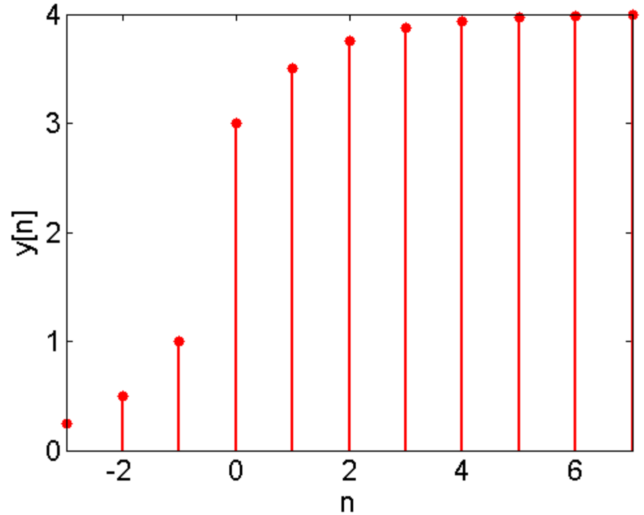
$$y[n] = (x_1[n] + x_2[n]) * h[n]$$

and $y[n] = y_1[n]+y_2[n]$, where $y_1[n] = x_1[n]*h[n]$, $y_1[n] = x_1[n]*h[n]$.

From Lecture 3 - example 3, and O&W example 2.5

$$y_1[n] = \left(\frac{1-0.5^{n+1}}{1-0.5}\right)u[n]$$

$$y_2[n] = \begin{cases} 2^{n+1} & n \le 0 \\ 2 & n \ge 1 \end{cases}$$



Associative Property (Serial Systems)

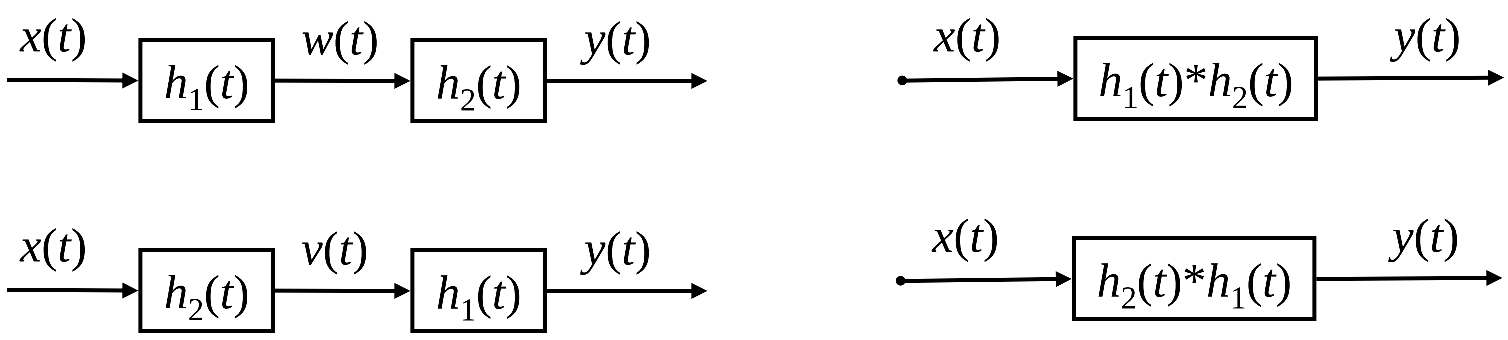
Another property of (LTI) convolution is that it is associative

$$x[n]*(h_1[n]*h_2[n]) = (x[n]*h_1[n])*h_2[n]$$

$$x(t)*(h_1(t)*h_2(t)) = (x(t)*h_1(t))*h_2(t)$$

Again this can be easily verified by manipulating the summation/integral indices

Therefore, the following four systems are all equivalent and $y[n] = x[n]*h_1[n]*h_2[n]$ is unambiguously defined.



This is not true for non-linear systems ($y_1[n] = 2x[n]$, $y_2[n] = x^2[n]$)

LTI System Memory

An LTI system is memoryless if its output depends only on the input value at the same time, i.e.

$$y[n] = kx[n]$$

$$y(t) = kx(t)$$

For an impulse response, this can only be true if

$$h[n] = k\delta[n]$$

$$h(t) = k\delta(t)$$

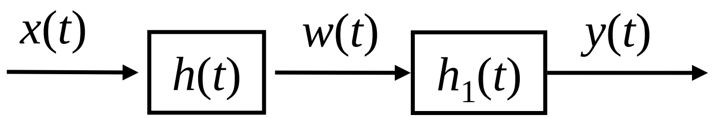
Such systems are extremely simple and the output of dynamic engineering, physical systems depend on:

- Preceding values of $x[n-1]$, $x[n-2]$, …
- Past values of $y[n-1]$, $y[n-2]$, …

for discrete-time systems, or derivative terms for continuous-time systems

System Invertibility

Does there exist a system with impulse response $h_1(t)$ such that $y(t)=x(t)$?



Widely used concept for:

- **control** of physical systems, where the aim is to calculate a control signal such that the system behaves as specified
- **filtering** out noise from communication systems, where the aim is to recover the original signal $x(t)$

The aim is to calculate "inverse systems" such that

$$h[n]h_1[n] = \delta[n]$$

$$h(t)h_1(t) = \delta(t)$$

The resulting **serial** system is therefore **memoryless**

Example: Accumulator System

Consider a DT LTI system with an impulse response

$h[n] = u[n]$

Using convolution, the response to an arbitrary input $x[n]$:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

As $u[n-k] = 0$ for $n-k<0$ and 1 for $n-k\geq 0$, this becomes

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

i.e. it acts as a running sum or accumulator. Therefore an inverse system can be expressed as:

$$y[n] = x[n] - x[n-1]$$

A first difference (differential) operator, which has an impulse response

$$h_1[n] = \delta[n] - \delta[n-1]$$

Causality for LTI Systems

Remember, a causal system only depends on present and past values of the input signal. We do not use knowledge about future information.

For a discrete LTI system, convolution tells us that

$h[n] = 0 \quad \text{for } n<0$

as $y[n]$ must not depend on $x[k]$ for $k>n$, as the impulse response must be zero before the pulse!

$$x[n]*h[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

$$x(t)*h(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

Both the integrator and its inverse in the previous example are causal

This is strongly related to inverse systems as we generally require our inverse system to be causal. If it is not causal, it is difficult to manufacture!

LTI System Stability

Remember: A system is stable if every bounded input produces a bounded output

Therefore, consider a bounded input signal

$|x[n]| < B$ for all n

Applying convolution and taking the absolute value:

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

Using the triangle inequality (magnitude of a sum of a set of numbers is no larger than the sum of the magnitude of the numbers):

$$|y[n]| \leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

$$\leq B \sum_{k=-\infty}^{\infty} |h[k]|$$

Therefore a DT **LTI system is stable** if and only if its impulse response is absolutely summable, ie

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Continuous-time system $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$

Example: System Stability

Are the DT and CT pure time shift systems stable?

$$h[n] = \delta[n - n_0]$$

$$h(t) = \delta(t - t_0)$$

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} |\delta[k - n_0]| = 1 < \infty$$

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_{-\infty}^{\infty} |\delta(\tau - t_0)| d\tau = 1 < \infty$$

Therefore, both the CT and DT systems are **stable**: all finite input signals produce a finite output signal

Are the discrete and continuous-time integrator systems stable?

$$h[n] = u[n - n_0]$$

$$h(t) = u(t - t_0)$$

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} |u[k - n_0]| = \sum_{k=n_0}^{\infty} |u[k]| = \infty$$

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_{-\infty}^{\infty} |u(\tau - t_0)| d\tau = \int_{t_0}^{\infty} |u(\tau)| d\tau = \infty$$

Therefore, both the CT and DT systems are **unstable**: at least one finite input causes an infinite output signal

Differential and Difference Equations

Two extremely important classes of causal LTI systems:

1) CT systems whose input-output response is described by **linear, constant-coefficient, ordinary differential equations** with a forced response

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

RC circuit with: $y(t) = v_c(t)$, $x(t) = v_s(t)$, $a = b = 1/RC$.

2) DT systems whose input-output response is described by **linear, constant-coefficient, difference equations**

$$y[n] + ay[n-1] = bx[n]$$

Simple bank account with: a = -1.01, b = 1.

Note that to "solve" these equations for $y(t)$ or $y[n]$, we need to know the initial conditions

Examine such systems and relate them to the system properties just described

Continuous-Time Differential Equations

A general N^{th}-order LTI differential equation is

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

If the equation involves derivative operators on $y(t)$ ($N>0$) or $x(t)$, it has memory.

The system stability depends on the coefficients a_k. For example, a 1st order LTI differential equation with $a_0=1$:

$$\frac{dy(t)}{dt} - a_1 y(t) = 0 \qquad\qquad y(t) = Ae^{a_1 t}$$

If $a_1>0$, the system is unstable as its impulse response represents a growing exponential function of time

If $a_1<0$ the system is stable as its impulse response corresponds to a decaying exponential function of time

Discrete-Time Difference Equations

A general N^{th}-order LTI difference equation is

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

If the equation involves difference operators on $y[n]$ ($N>0$) or $x[n]$, it has memory.

The system stability depends on the coefficients a_k. For example, a 1st order LTI difference equation with $a_0=1$:

$$y[n] - a_1 y[n-1] = 0 \qquad\qquad y[n] = A a_1^n$$

If $a_1>1$, the system is unstable as its impulse response represents a growing power function of time

If $a_1<1$ the system is stable as its impulse response corresponds to a decaying power function of time

Lecture 6: Summary

The standard notions of commutative, associative and distributive properties are valid for convolution operators. These can be used to simplify evaluating convolution, by decomposing the input system/signal into simpler parts, and then solving the transformed problem.

Standard system notations of

- Memory
- Causality
- Invertibility
- Stability

Can be given specific definitions in terms of representing a system via its convolution response or in terms of the derivative/differential equation