EE-232 Signals & Systems

Lecture 7

Basis Functions & Fourier Series

Asst Prof Kamran Aziz Bhatti

Basis Functions & Fourier Series

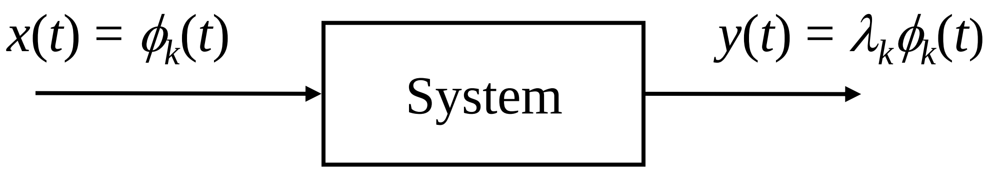
Basis functions (3 lectures): **Concept of basis function. Fourier series representation** of time functions. Fourier transform and its properties. Examples, transform of simple time functions.

Specific objectives for today:

- Introduction to Fourier series (& transform)
- Eigenfunctions of a system
  - Show sinusoidal signals are eigenfunctions of LTI systems
- Introduction to signals and basis functions
- Fourier basis & coefficients of a periodic signal

Why is Fourier Theory Important (i)?

For a particular system, what signals $\phi_k(t)$ have the property that:



Then $\phi_k(t)$ is an **eigenfunction** with **eigenvalue** λ_k

If an input signal can be decomposed as
$x(t) = \Sigma_k a_k \phi_k(t)$

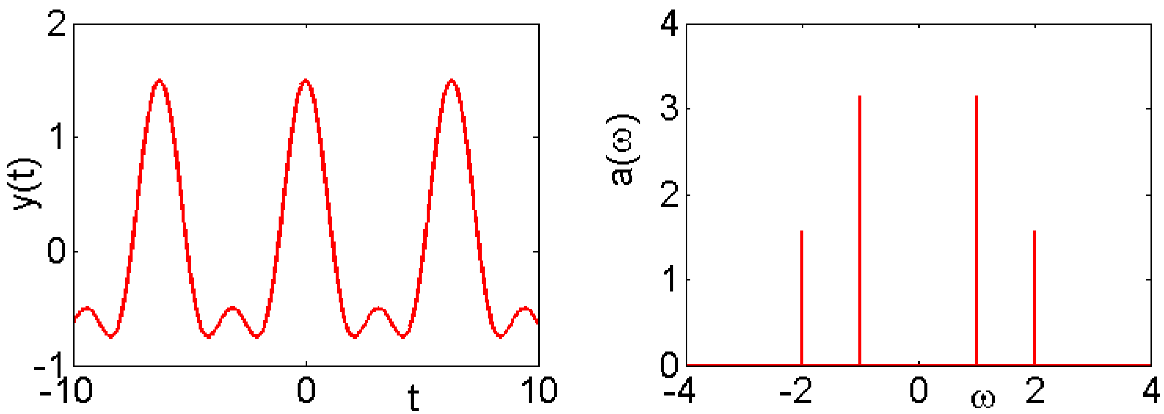
Then the response of an LTI system is
$y(t) = \Sigma_k a_k \lambda_k \phi_k(t)$

For an LTI system, $\phi_k(t) = e^{st}$ where $s \in C$, are eigenfunctions.

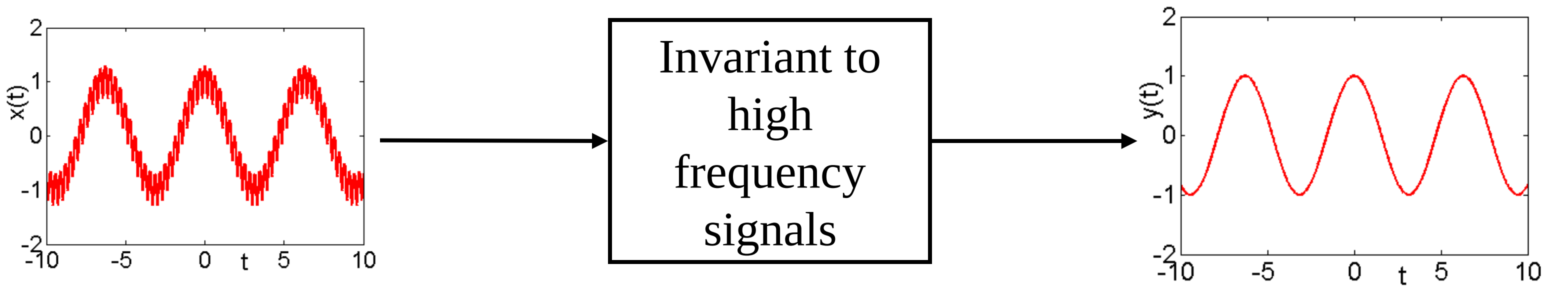
Why is Fourier Theory Important (ii)?

Fourier transforms map a time-domain signal into a frequency domain signal

Simple interpretation of the frequency content of signals in the frequency domain (as opposed to time).



Design systems to filter out high or low frequency components. Analyse systems in frequency domain.



Why is Fourier Theory Important (iii)?

If $F\{x(t)\} = X(j\omega)$ ω is the frequency

Then $F\{x'(t)\} = j\omega X(j\omega)$

So solving a differential equation is transformed from a **calculus operation** in the time domain into an **algebraic operation** in the frequency domain (see Laplace transform)

Example $$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = 0$$

becomes $$-\omega^2 Y(j\omega) + j2\omega Y(j\omega) + 3Y(j\omega) = 0$$

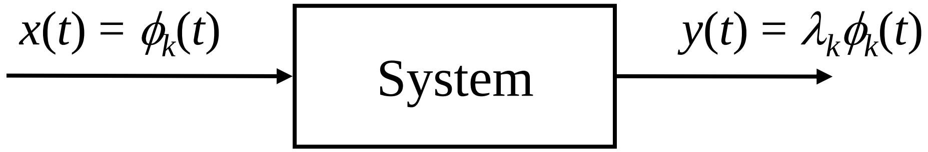
and is solved for the roots ω:

$$-\omega^2 + j2\omega + 3 = 0$$

and we take the inverse Fourier transform for those ω.

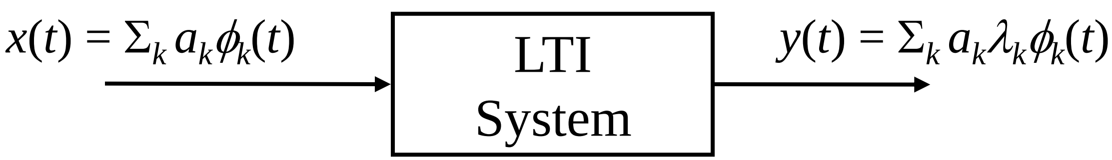
Introduction to System Eigenfunctions

Lets imagine what (basis) signals $\phi_k(t)$ have the property that:



i.e. the output signal is the same as the input signal, multiplied by the constant "gain" λ_k (which may be complex)

For CT LTI systems, we also have that



Therefore, to make use of this theory we need:

1) **system identification** is determined by finding $\{\phi_k, \lambda_k\}$.

2) **response**, we also have to decompose $x(t)$ in terms of $\phi_k(t)$ by calculating the coefficients $\{a_k\}$.

This is analogous to eigenvectors/eigenvalues matrix decomposition

Complex Exponentials are Eigenfunctions of any CT LTI System

Consider a CT LTI system with impulse response $h(t)$ and input signal $x(t)=\phi(t) = e^{st}$, for any value of $s \in C$:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{st}e^{-s\tau}d\tau \\ &= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \end{aligned}$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

Assuming that the integral on the right hand side converges to $H(s)$, this becomes (for any value of $s \in C$):

$$y(t) = H(s)e^{st}$$

Therefore $\phi(t)=e^{st}$ is an eigenfunction, with eigenvalue $\lambda=H(s)$

Example 1: Time Delay & Imaginary Input

Consider a CT, LTI system where the input and output are related by a pure time shift:

$$y(t) = x(t-3)$$

Consider a purely imaginary input signal:

$$x(t) = e^{j2t}$$

Then the response is:

$$y(t) = e^{j2(t-3)} = e^{-j6}e^{j2t}$$

e^{j2t} is an eigenfunction (as we'd expect) and the associated eigenvalue is $H(j2) = e^{-j6}$.

The eigenvalue could be derived "more generally". The system impulse response is $h(t) = \delta(t\text{-}3)$, therefore:

$$H(s) = \int_{-\infty}^{\infty} \delta(\tau - 3)e^{-s\tau}d\tau = e^{-3s}$$

So $H(j2) = e^{-j6}$!

Example 1a: Phase Shift

Note that the corresponding input e^{-j2t} has eigenvalue e^{j6}, so lets consider an input cosine signal of frequency 2 so that:

$$\cos(2t) = \tfrac{1}{2}\left(e^{j2t} + e^{-j2t}\right)$$

By the system LTI, eigenfunction property, the system output is written as:

$$\begin{aligned} y(t) &= \tfrac{1}{2}\left(e^{-j6}e^{j2t} + e^{j6}e^{-j2t}\right) \\ &= \tfrac{1}{2}\left(e^{j(2t-6)} + e^{-j(2t-6)}\right) \\ &= \cos(2(t-3)) \end{aligned}$$

So because the eigenvalue is **purely imaginary**, this corresponds to a **phase shift** (time delay) in the system's response. If the eigenvalue had a real component, this would correspond to an amplitude variation

Example 2: Time Delay & Superposition

Consider the same system (3 time delays) and now consider the input signal $x(t) = \cos(4t)+\cos(7t)$, a superposition of two sinusoidal signals that are not harmonically related. The response is obviously:

$$y(t) = \cos(4(t-3)) + \cos(7(t-3))$$

Consider $x(t)$ represented using Euler's formula:

$$x(t) = \tfrac{1}{2}e^{j4t} + \tfrac{1}{2}e^{-j4t} + \tfrac{1}{2}e^{j7t} + \tfrac{1}{2}e^{-j7t}$$

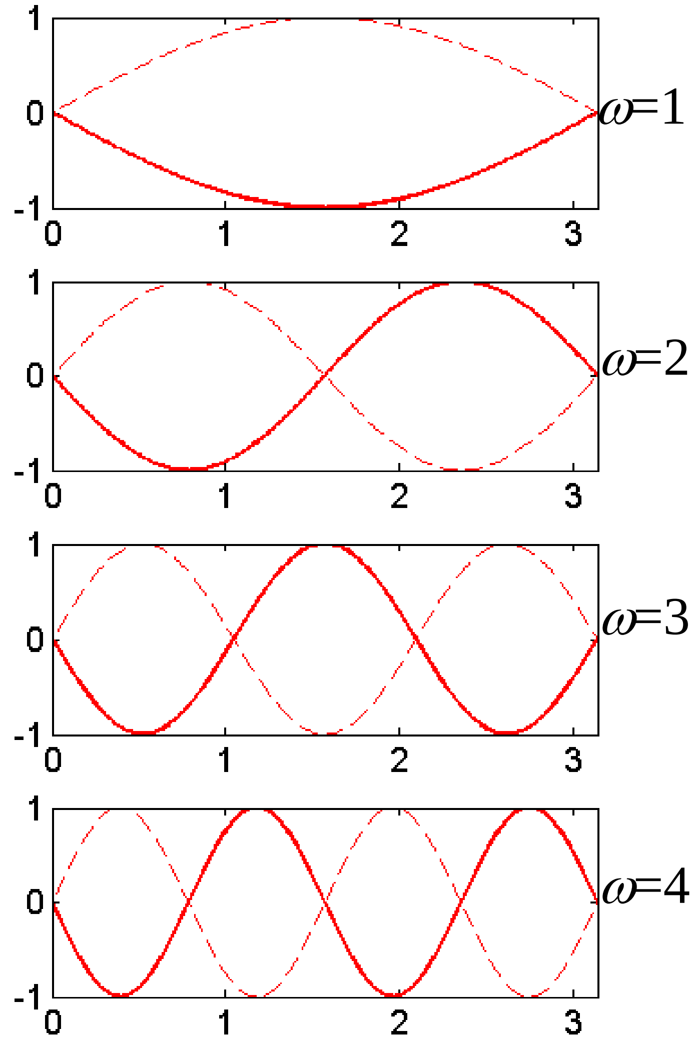
Then due to the superposition property and $H(s) = e^{-3s}$

$$\begin{aligned} y(t) &= \tfrac{1}{2}e^{-j12}e^{j4t} + \tfrac{1}{2}e^{j12}e^{-j4t} + \tfrac{1}{2}e^{-j21}e^{j7t} + \tfrac{1}{2}e^{j21}e^{-j7t} \\ &= \tfrac{1}{2}e^{j4(t-3)} + \tfrac{1}{2}e^{-j4(t-3)} + \tfrac{1}{2}e^{j7(t-3)} + \tfrac{1}{2}e^{-j7(t-3)} \\ &= \cos(4(t-3)) + \cos(7(t-3)) \end{aligned}$$

While the answer for this simple system can be directly spotted, the superposition property allows us to apply the eigenfunction concept to more complex LTI systems.

History of Fourier/Harmonic Series

- The idea of using trigonometric sums was used to predict astronomical events
- Euler studied vibrating strings, ~1750, which are signals where linear displacement was preserved with time.
- Fourier described how such a series could be applied and showed that a periodic signals can be represented as the integrals of sinusoids that are not all harmonically related
- Now widely used to understand the structure and frequency content of arbitrary signals



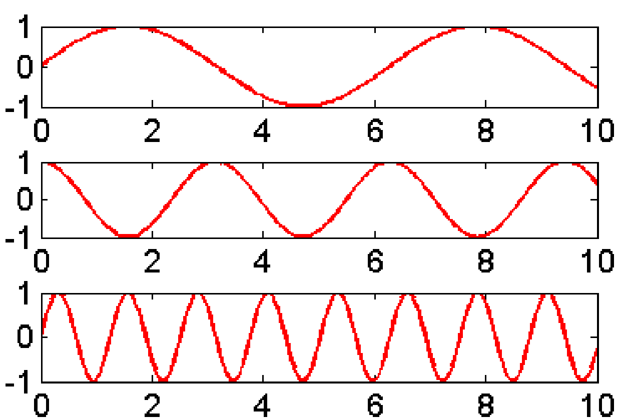
Fourier Series and Fourier Basis Functions

The theory derived for LTI convolution, used the concept that any input signal can represented as a linear combination of shifted impulses (for either DT or CT signals)

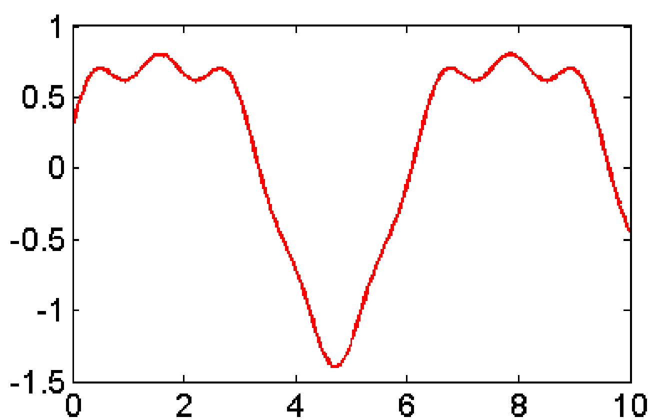
We will now look at how (input) signals can be represented as a linear combination of **Fourier basis functions** (LTI eigenfunctions) which are purely imaginary exponentials

These are known as continuous-time **Fourier series**

The bases are **scaled and shifted sinusoidal signals**, which can be represented as **complex exponentials**



$e^{j\omega t}$



$x(t)$

$x(t) = \sin(t) + 0.2\cos(2t) + 0.1\sin(5t)$

Periodic Signals & Fourier Series

A periodic signal has the property $x(t) = x(t+T)$, T is the fundamental period, $\omega_0 = 2\pi/T$ is the fundamental frequency. Two periodic signals include:

$$x(t) = \cos(\omega_0 t)$$

$$x(t) = e^{j\omega_0 t}$$

For each periodic signal, the **Fourier basis** the set of **harmonically related complex exponentials**:

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t} \qquad k = 0, \pm 1, \pm 2, \ldots$$

Thus the **Fourier series** is of the form:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

k=0 is a constant

k=+/-1 are the fundamental/first harmonic components

k=+/-N are the N^{th} harmonic components

For a particular signal, are the values of $\{a_k\}_k$?

Fourier Series Representation of a CT Periodic Signal (i)

Given that a signal has a Fourier series representation, we have to find $\{a_k\}_k$. Multiplying through by $e^{-jn\omega_0 t}$

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}dt$$

T is the fundamental period of $x(t)$

$$= \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}dt$$

Using Euler's formula for the complex exponential integral

$$\int_0^T e^{j(k-n)\omega_0 t}dt = \int_0^T \cos((k-n)\omega_0 t)dt + j\int_0^T \sin((k-n)\omega_0 t)dt$$

It can be shown that

$$\int_0^T e^{j(k-n)\omega_0 t}dt = \begin{cases} T & k=n \\ 0 & k \neq n \end{cases}$$

Fourier Series Representation of a CT Periodic Signal (ii)

Therefore

$$a_n = \tfrac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$$

which allows us to determine the coefficients. Also note that this result is the same if we integrate over any interval of length *T* (not just [0,*T*]), denoted by $\int_T$

$$a_n = \tfrac{1}{T}\int_T x(t)e^{-jn\omega_0 t}dt$$

To summarise, if x(t) has a Fourier series representation, then the pair of equations that defines the Fourier series of a periodic, continuous-time signal:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

$$a_k = \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \tfrac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

Lecture 7: Summary

Fourier bases, series and transforms are extremely useful for frequency domain analysis, solving differential equations and analysing invariance for LTI signals/systems

For an LTI system

- e^{st} is an **eigenfunction**
- $H(s)$ is the corresponding (complex) **eigenvalue**

This can be used, like convolution, to calculate the output of an LTI system once $H(s)$ is known.

A **Fourier basis** is a set of harmonically related complex exponentials

Any periodic signal can be represented as an infinite sum (**Fourier series**) of Fourier bases, where the first harmonic is equal to the fundamental frequency

The corresponding coefficients can be evaluated

Lecture 7: Exercises

SaS, O&W, Q3.1-3.5