Digital Signal Processing

Z-transform

dftwave

z-Transform

Background-Definition

- Fourier transform $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$ extracts the essence of x[n]

  but is limited in the sense that it can handle stable systems only.

  $X(e^{j\omega})$ converges if $\sum |x[n]| < \infty$

  i.e., stable system $\rightarrow$ Fourier Transform converges

- So, we want to extend it such that it can be used as a tool to analyze digital systems in general.

Let $X_r(e^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}$

then it converges if $\sum |x[n]r^{-n}| < \infty$

The condition for convergence is relaxed!

(e.g.) $x[n] = 2^n u[n]$

$$|X(e^{j\omega})| = |\sum 2^n e^{-j\omega n}| \to \infty$$

$$|X_r(e^{j\omega})| = |\sum 2^n r^{-n} e^{-j\omega n}| < \sum 2^n r^{-n} = \frac{1}{1 - \frac{2}{|r|}}$$

$\to$ converges if $|r| > 2$

- This implies that $X_r(e^{j\omega})$ can handle some systems that $X(e^{j\omega})$ cannot due to divergence.

- Therefore we define z-transform to be

$$X(z) = X_r(e^{j\omega})|_{re^{jw}=z} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Representing the condition for convergence of $X_r(e^{j\omega})$

in terms of region of convergence RoC.

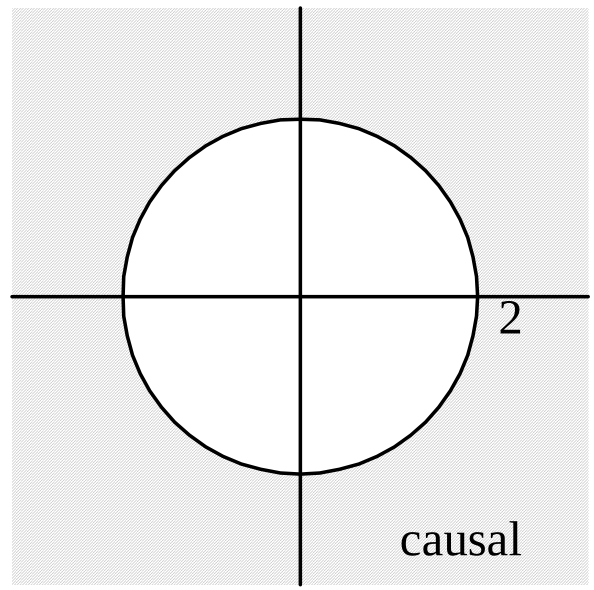
(e.g.) in case $x[n] = 2^n u[n]$

$X_r(e^{j\omega})$ exists for $|r|>2$.

So, RoC is $|z| = |re^{j\omega}|>2$.

In general, if $x[n] = a^n u[n]$

$R_o C$ is $|z| > |a|$



- In terms of $X(z)$,

$X(e^{jw})$ is a special case

Where $|z| = 1$, or $r = 1$

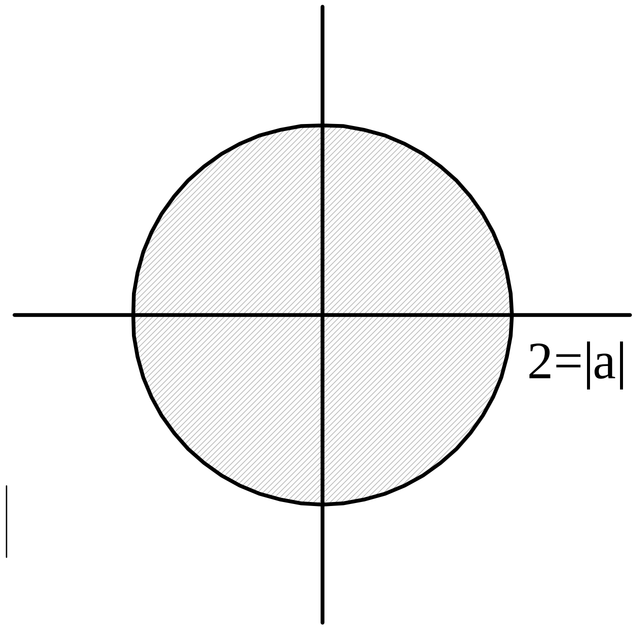
(e.g.) $x[n] = -a^n u[-n-1]$

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n$$

$$= 1 - \frac{1}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z}$$

$$= \frac{1}{1-az^{-1}}$$



$R_o C$: $|a^{-1} z| < 1$, or $|z| < |a|$

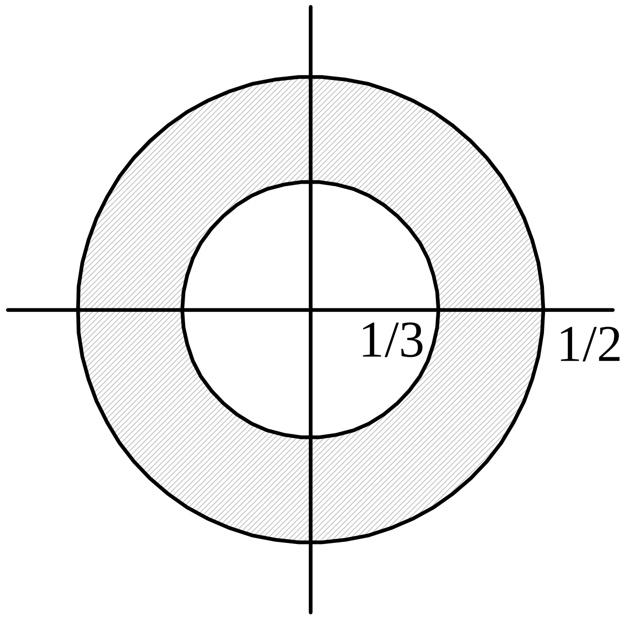
(e.g.) Two - sided sequence

$$x[n] = (-\frac{1}{3})^n u[n] - (\frac{1}{2})^n u[-n-1]$$

$$X(z) = \sum_{n=0}^{\infty} (-\frac{1}{3})^n z^{-n} - \sum_{n=-\infty}^{-1} (\frac{1}{2})^n z^{-n}$$

$$= \frac{1}{1+\frac{1}{3}z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$|z| > \frac{1}{3} \ , \qquad |z| < \frac{1}{2}$$



Some Common z-Transforms

(1)	$\delta[n]$	$\leftrightarrow$	1	$[all \;\; z]$
(2)	$u[n]$	$\leftrightarrow$	$\dfrac{1}{1-z^{-1}}$	$[\lvert z\rvert > 1]$
(3)	$-u[-n-1]$	$\leftrightarrow$	$\dfrac{1}{1-z^{-1}}$	$[\lvert z\rvert < 1]$
(4)	$\delta[n-m]$	$\leftrightarrow$	z^{-m}	$[all \;\; z \;\; except \;\; 0, \; if \;\; m > 0,$ $all \;\; z \;\; except \;\; \infty, \; if \;\; m < 0]$
(5)	$a^n u[n]$	$\leftrightarrow$	$\dfrac{1}{1-az^{-1}}$	$[\lvert z\rvert > \lvert a\rvert]$
(6)	$-a^n u[-n-1]$	$\leftrightarrow$	$\dfrac{1}{1-az^{-1}}$	$[\lvert z\rvert < \lvert a\rvert]$
(7)	$na^n u[n]$	$\leftrightarrow$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$[\lvert z\rvert > \lvert a\rvert]$

(8) [illegible] $\longleftrightarrow$ [illegible] [illegible]

(9) [illegible] $\longleftrightarrow$ [illegible] [illegible]

(10) [illegible] $\longleftrightarrow$ [illegible] [illegible]

(11) [illegible] $\longleftrightarrow$ [illegible] [illegible]

(12) [illegible] $\longleftrightarrow$ [illegible] [illegible]

(13) [illegible] $\longleftrightarrow$ [illegible] [illegible]

Properties of R_oC

(1) in general $0 \le r_R < R_oC < r_L \le \infty$

(2) $X(e^{jw})$ absolutely converges $\leftrightarrow UC \subset R_oC$

(3) R_oC cannot contain a pole

(4) FIR sequence $\rightarrow$ entire z plane, may be except for 0 or ∞

(5) Right-sided sequence $\rightarrow$ outward of the outermost pole

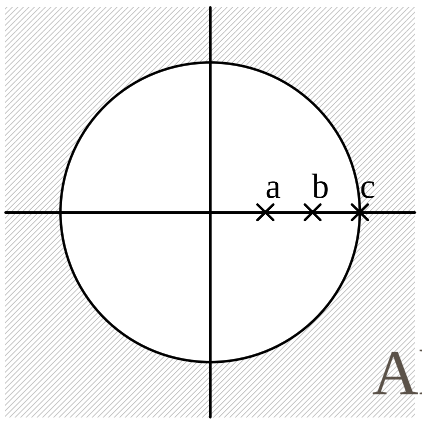
(6) Left-sided sequence $\rightarrow$ inward from the innermost pole

(7) Two-sided sequence $\rightarrow$ a ring in between two adjacent rings

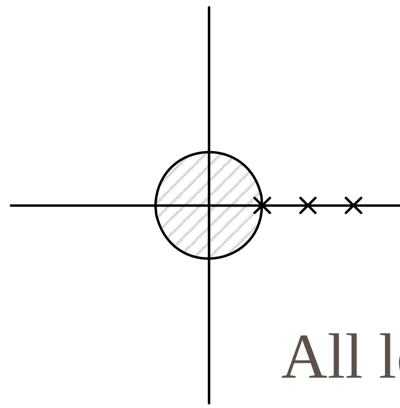
(8) R_oC is a connected region

(e.g.) If x[n] is a sum of 3 sequences whose poles are a, b, c respectively,

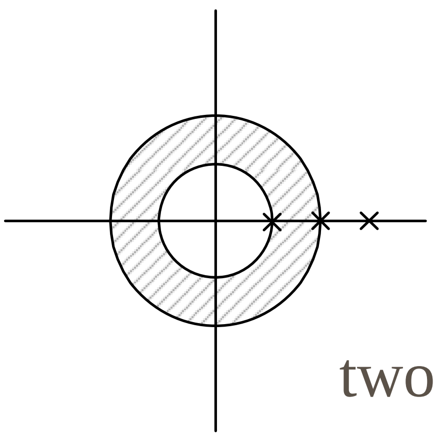
There exist A possible R_oCs as shown below



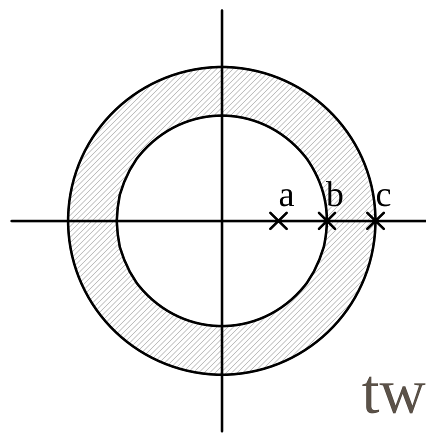
All right-sided



All left-sided



two left-sided



two right-sided

z–Transform Properties

(1) Linearity

$$a x_1[n] + b x_2[n] \quad \leftrightarrow \quad a X_1(z) + b X_2(z)$$

(2) Time shifting

$$x[n-n_o] \quad \leftrightarrow \quad z^{-n_o} X(z)$$

(e.g.)

$$\frac{z^{-1}}{1-\frac{1}{4}z^{-1}} \leftrightarrow z^{-1} \cdot \frac{1}{1-\frac{1}{4}z^{-1}} \rightarrow (\frac{1}{4})^n u[n] \quad \text{and delay -1}$$

$$\rightarrow (\frac{1}{4})^{n-1} u[n-1], \quad |z| > \frac{1}{4}$$

z–Transform Properties..(cont.)

(3) Multiplication by an Exponential Sequence

$$z_o^n x[n] \leftrightarrow X(\frac{z}{z_o})$$

(e.g.) $$(e^{jw_o})^n x[n] \leftrightarrow X(e^{-jw_o} \cdot z) \xrightarrow{z = e^{jw}} X(e^{j(w-w_o)})$$

(e.g.) $$r^n \cos w_o n \cdot u[n] = \frac{r^n}{2}[(e^{jw_o})^n + (e^{-jw_o})^n]u[n]$$

$$= \frac{1}{2}[(re^{jw_o})^n + (re^{-jw_o})^n]u[n]$$

$$\leftrightarrow \frac{1}{2}[\frac{1}{1 - re^{jw} \cdot z^{-1}} + \frac{1}{1 - re^{-jw_o} z^{-1}}]$$

$$= \frac{1 - r\cos w_o z^{-1}}{1 - 2r\cos w_o z^{-1} + r^2 z^{-2}} \qquad |z| > r$$

z–Transform Properties..(cont.)

(4) Differentiation of X(z)

$$nx[n] \leftrightarrow -z\frac{d}{dz}x[z] \qquad R_oC = R_x$$

(e.g.) $X(z) = \log(1+az^{-1}) \qquad |z| > |a|$

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1+az^{-1}}$$

$$-z\frac{d}{dz}X(z) = \frac{az^{-1}}{1+az^{-1}} = a \cdot z^{-1} \cdot \frac{1}{1+az^{-1}}$$

$$nx[n] = a(-a)^{n-1}u[n-1]$$

$$x[n] = (-1)^{n-1}\frac{a^n}{n}u[n-1]$$

z–Transform Properties..(cont.)

(5) Conjugation of Complex Sequence

$$x^*[n] \leftrightarrow X^*(z^*) \qquad R_oC = R_X$$

(6) Time-Reversal

$$x^*[-n] \leftrightarrow X^*(\frac{1}{z^*}) \qquad R_oC = \frac{1}{R_X}$$

$$x[-n] \leftrightarrow X(\frac{1}{z})$$

(7) Convolution-Integration

$$x_1[n] * x_2[n] \leftrightarrow X_1(z) X_2(z)$$

Inverse z–Transform

4-Ways:

- By the contour integral of the inverse transform definition
- "By Inspection" - recognize common transform pairs
- Partial Fraction Expansion - effective for rational z-transforms
- Power Series Expansion

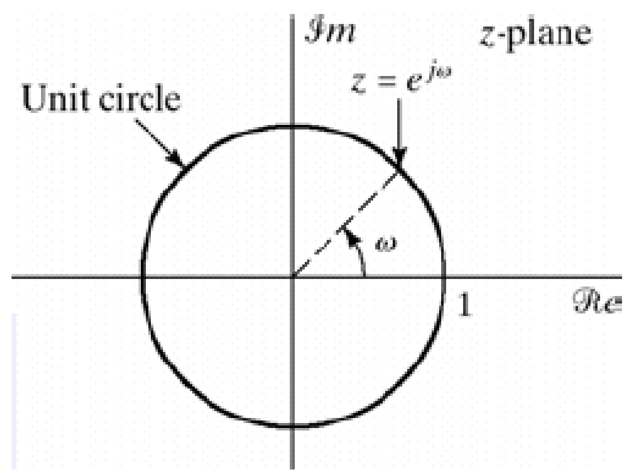
Inversion by Contour Integration

- Cauchy integral definition of the inverse z-Transform

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- Example: Inverse DTFT $z = e^{j\omega}$
- Implies.. Contour C is chosen as unit circle

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$



Inversion Method -1

Concept of Partial Fraction Expansion-

- Consider a general rational z-transform (no repeated roots); two equivalent forms are:

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

Inversion Method

Concept of Partial Fraction Expansion Inversion-

1. Find partial fraction expansion method in third equivalent form

$$X(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{only if } M \geq N} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

2. Invert by expansion

4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1 - az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1 - az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$

Doing the Partial Fraction Expansion

To find the coefficients of the m^{th} first-order pole, consider

$$(1-d_m z^{-1})X(z) = (1-d_m z^{-1})\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right] + \sum_{k=1}^{N} \frac{(1-d_m z^{-1})A_k}{1-d_k z^{-1}}$$

Now evaluate at $z = d_m$ ...

Doing the Partial Fraction Expansion–2

Thus we now have:

$$\left.(1-d_m z^{-1})X(z)\right|_{z=d_m} = (1-\cancelto{0}{d_m z^{-1}})\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]_z + \left.\sum_{k=1}^{N} \frac{(1-\cancelto{0}{d_m z^{-1}})A_k}{1-d_k z^{-1}}\right|_{z=d_m} = A_m$$

$$A_k = \left.(1-d_k z^{-1})X(z)\right|_{z=d_k}$$

We will see that this formula is applied to the product form in practice …

Writing Down x[n]

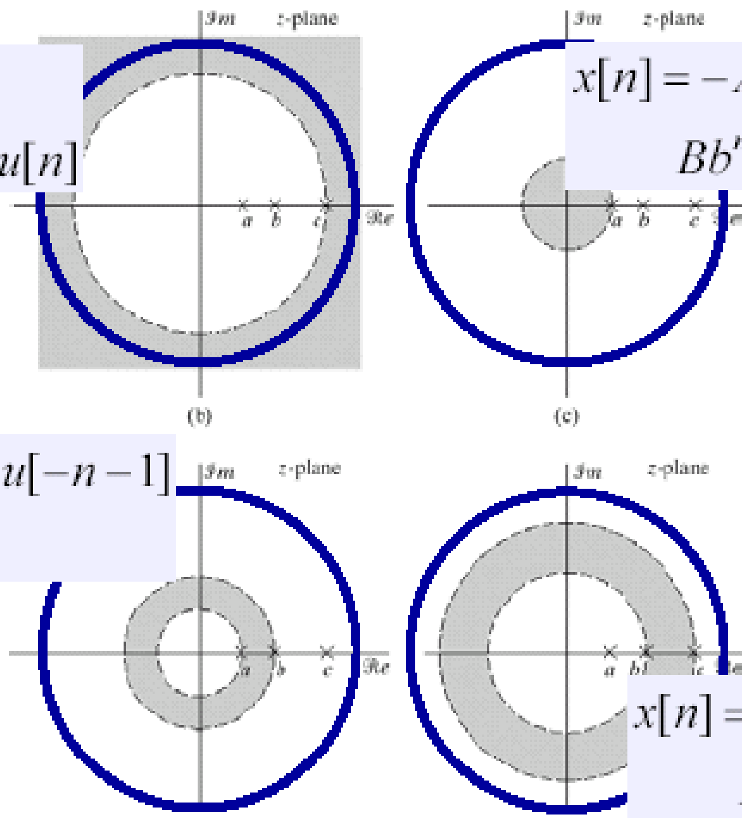
$$X(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{if } M \geq N} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

$$x[n] = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r \delta[n-r]\right]}_{\text{if } M \geq N}$$

$$+ \underbrace{\sum_{k} A_k d_k^n u[n]}_{\text{when } |d_k| < r_R} - \underbrace{\sum_{k} A_k d_k^n u[-n-1]}_{\text{when } |d_k| > r_L}$$

X[n] depend on knowing the ROC

$$X(z) = \frac{A}{1-az^{-1}} + \frac{B}{1-bz^{-1}} + \frac{C}{1-cz^{-1}}$$



$$x[n] = Aa^n u[n] + Bb^n u[n] + Cc^n u[n]$$

$$x[n] = -Aa^n u[-n-1] - Bb^n u[-n-1] - Cc^n u[-n-1]$$

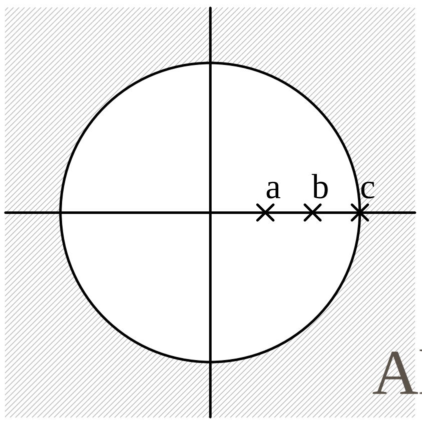
$$x[n] = Aa^n u[n] - Bb^n u[-n-1] - Cc^n u[-n-1]$$

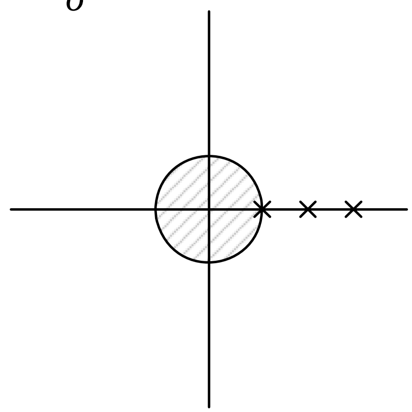
$$x[n] = Aa^n u[n] + Bb^n u[n] - Ca^n u[-n-1]$$

Example– ROC

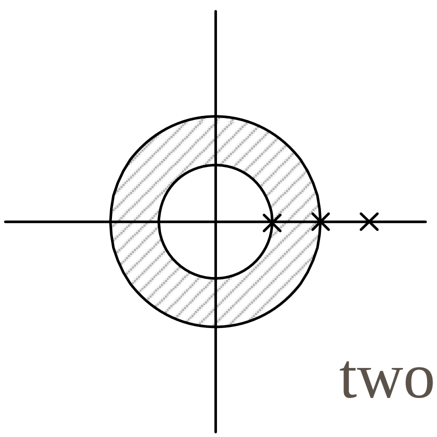
If x[n] is a sum of 3 sequences whose poles are a, b, c respectively,

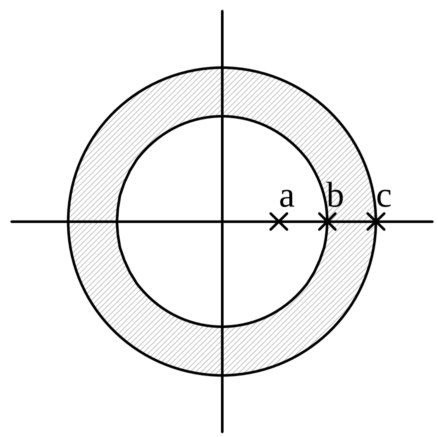
There exist A possible R_oCs as shown below





All right-sided





two left-sided

Example– Partial Fraction

$$X(z) = \frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-3z^{-1}+2z^{-2}} = \frac{(1+z^{-1})^3}{(1-z^{-1})(1-2z^{-1})}$$

Long Division

$$X(z) = \frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-3z^{-1}+2z^{-2}} = \frac{(1+z^{-1})^3}{(1-z^{-1})(1-2z^{-1})}$$

$$\begin{array}{r|l} & 0.5z^{-1} + 2.25 \\ \hline 2z^{-2} - 3z^{-1} + 1 & z^{-3} + 3.0z^{-2} + 3.0z^{-1} + 1.0 \\ & \underline{z^{-3} - 1.5z^{-2} + 0.5z^{-1}} \\ & 4.5z^{-2} + 2.50z^{-1} \\ & \underline{4.5z^{-2} - 6.75z^{-1} + 2.25} \\ & 9.25z^{-1} - 1.25 \end{array}$$

$X(z) =$

$$2.25 + 0.5z^{-1} - \frac{1.25 - 9.25z^{-1}}{1-3z^{-1}+2z^{-2}} = 2.25 + 0.5z^{-1} - \frac{1.25 - 9.25z^{-1}}{(1-z^{-1})(1-2z^{-1})}$$

Finding the coefficients of Poles

$$X(z) = \frac{(1+z^{-1})^3}{(1-z^{-1})(1-2z^{-1})}$$

$$= 2.25 + 0.5z^{-1} + \frac{A_1}{(1-z^{-1})} + \frac{A_2}{(1-2z^{-1})}$$

$$A_1 = X(z)(1-z^{-1})\Big|_{z=1} = \left.\frac{(1+z^{-1})^3}{(1-2z^{-1})}\right|_{z=1} = \frac{8}{-1} = -8$$

$$A_2 = X(z)(1-2z^{-1})\Big|_{z=2} = \left.\frac{(1+z^{-1})^3}{(1-z^{-1})}\right|_{z=2} = \frac{(3/2)^3}{1/2} = 6.75$$

$$X(z) = 2.25 + 0.5z^{-1} + \frac{-8}{(1-z^{-1})} + \frac{6.75}{(1-2z^{-1})}$$

Writing Down x[n]

$$X(z) = 2.25 + 0.5z^{-1} + \frac{-8}{(1 - z^{-1})} + \frac{6.75}{(1 - 2z^{-1})}$$

- If ROC is $2 < |z|$

$$x[n] = 2.25\delta[n] + 0.5\delta[n-1] - 8u[n] + 6.75(2)^n u[n]$$

- If ROC is $1 < |z| < 2$

$$x[n] = 2.25\delta[n] + 0.5\delta[n-1] - 8u[n] - 6.75(2)^n u[-n-1]$$

- If ROC is $|z| < 1$

$$x[n] = 2.25\delta[n] + 0.5\delta[n-1] + 8u[-n-1] - 6.75(2)^n u[-n-1]$$

Partial Fraction Expansion in MATLAB

$$X(z)=\frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-3z^{-1}+2z^{-2}}=2.25+0.5z^{-1}+\frac{-8}{1-z^{-1}}+\frac{6.75}{1-2z^{-1}}$$

```
» [r,p,k]=residuez([1,3,3,1],[1,-3,2])
r =
    6.75000000000000
   -8.00000000000000
p =
     2
     1
k =
    2.25000000000000    0.50000000000000
```

MATLAB's `residuez` *can also go back the other way, and can also handle repeated roots*

– *Note: '*`residuez`*' is part of the optional Signal Processing Toolbox*

Selected z-Transform Theorems

- The delay or shift property:

$$x[n-n_0] \quad \Leftrightarrow \quad z^{-n_0} X(z)$$

- The convolution property:

$$y[n] = x[n] * h[n] \quad \Leftrightarrow \quad Y(z) = H(z) X(z)$$

- A consequence:

$$y[n] = x[n] * \delta[n-n_0] \quad \Leftrightarrow \quad Y(z) = z^{-n_0} X(z)$$

An IIR System

- Difference equation:

$$y[n] = ay[n-1] + x[n] \Leftrightarrow Y(z) = az^{-1}Y(z) + X(z)$$

- System function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \quad |z|>|a|$$

ROC if causal

- Frequency response:

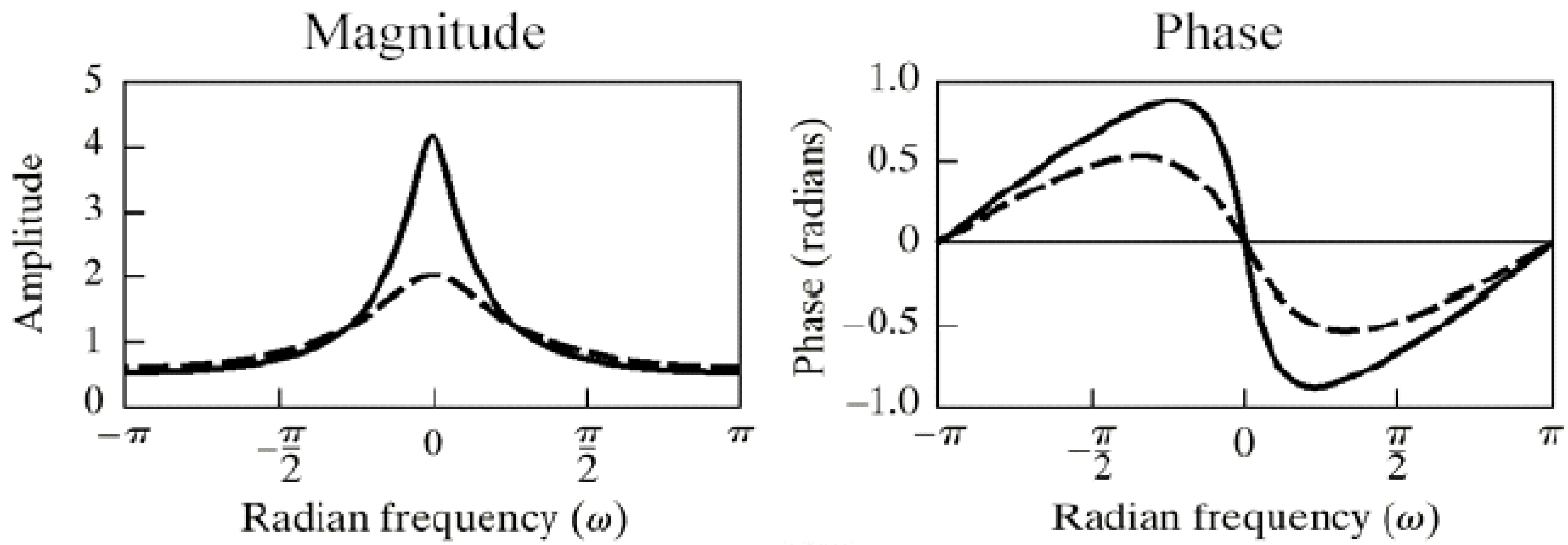
$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \frac{1}{1-ae^{-j\omega}}$$

Assumes $|a|<1$

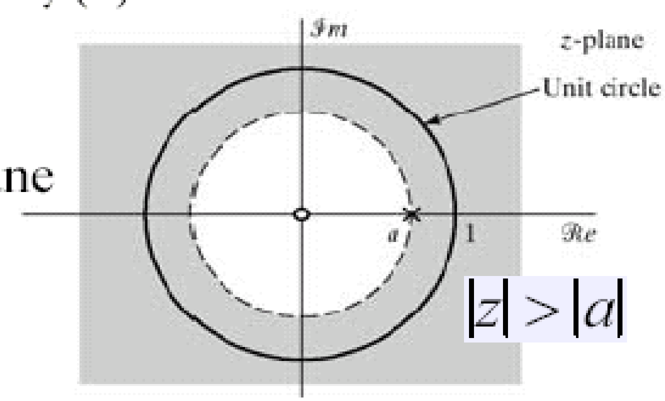
- Impulse response:

$$h[n] = a^n u[n]$$

IIR Frequency Response



z-plane



$|z| > |a|$

System Function Of a Difference Equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$\left(\sum_{k=0}^{N} a_k z^{-k} \right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k} \right) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left(\sum_{k=0}^{M} b_k z^{-k} \right)}{\left(\sum_{k=0}^{N} a_k z^{-k} \right)}$$

Difference equations give rise to rational z transforms!

H[*z*] and h[n]

- Consider a causal system; *i.e.* $h[n]=0$ for $n<0$:

$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=\frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

ROC:

$$|z|>\max_k |d_k|$$

$$H(z)=\underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{only if } M\geq N}+\sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}}$$

$$h[n]=\sum_{r=0}^{(M-N)} B_r \delta[n-r]+\sum_{k=1}^{N} A_k d_k^n u[n]$$

Frequency Response of a DE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \frac{\left(\sum_{k=0}^{M} b_k e^{-j\omega k}\right)}{\left(\sum_{k=0}^{N} a_k e^{-j\omega k}\right)}$$

ROC must Contain the Unit circle

LTI System Characterization

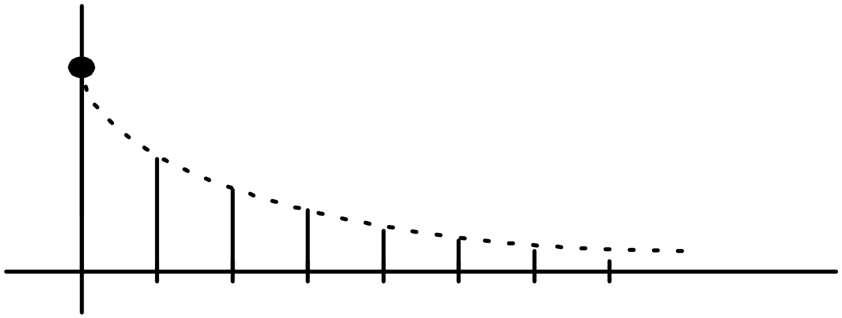
Input	Output
$x[n]$	$y[n]$
$\delta[n]$	$h[n]$
$e^{j\omega n}$	$H(e^{j\omega})e^{j\omega n}$
$X(e^{j\omega})$	$X(e^{j\omega})H(e^{j\omega})$
$X(z)$	$X(z)H(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left(\sum_{k=0}^{M} b_k z^{-k}\right)}{\left(\sum_{k=0}^{N} a_k z^{-k}\right)}$$

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

Stability, Causality– illustration

(1) $x[n] = (\frac{1}{2})^n u[n]$

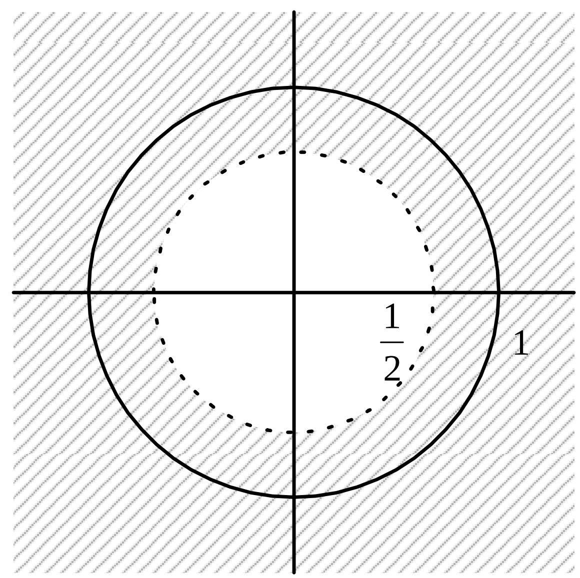


$$X(z) = \sum_{n=0}^{\infty} (\frac{1}{2})^n z^{-n} = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

$\text{RoC} : |z| > \frac{1}{2}$

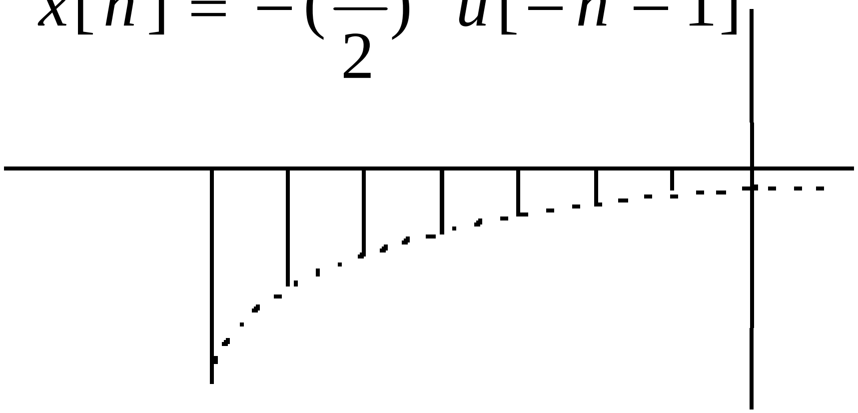
① Outward
② UC $\subset$ RoC

① Causal
② . Stable



Stability, Causality– illustration..(cont)

(2) $x[n] = -(\frac{1}{2})^n u[-n-1]$



① Anti Causal

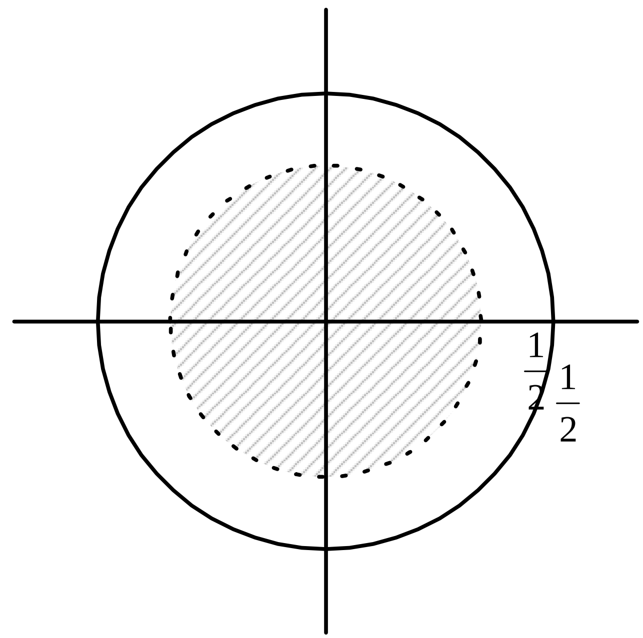
② Unstable

$$X(z) = \sum_{n=-\infty}^{-1} -(\frac{1}{2})^n z^{-n} = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$\text{ROC}: |z| < \frac{1}{2}$

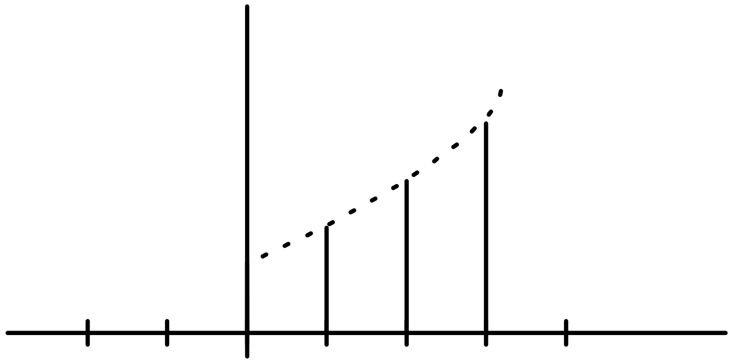
① Inward

② UC $\not\subset$ RoC



Stability, Causality– illustration..(cont)

(3) $x[n] = (2)^n u[n]$



① Causal

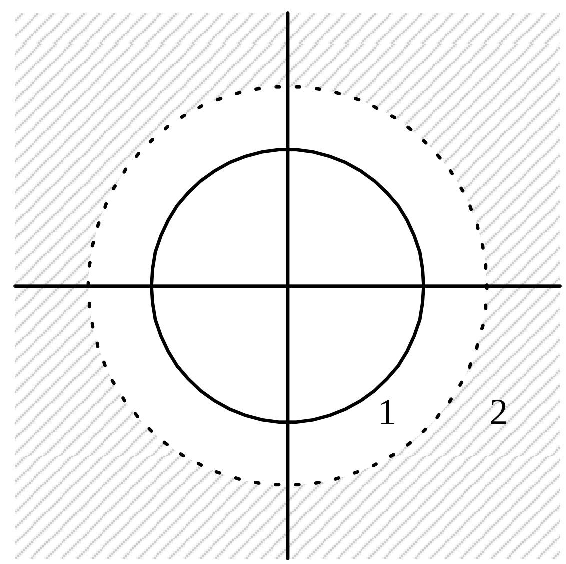
② Unstable

$$X(z) = \sum_{n=0}^{\infty} 2^n z^{-n} = \frac{1}{1-2z^{-1}}$$

- RoC : $|z| > 2$

① Outward

② UC $\not\subset$ RoC

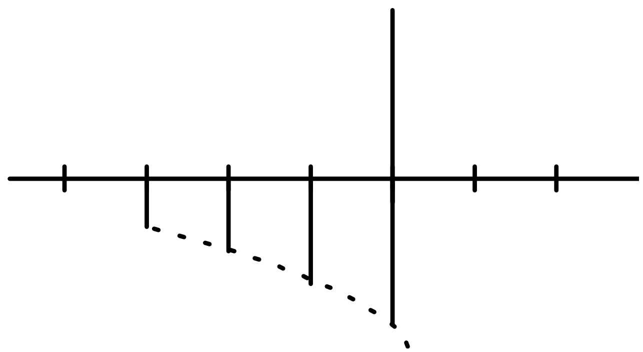


Stability, Causality– illustration..(cont)

(4) $x[n] = -2^n u[-n-1]$

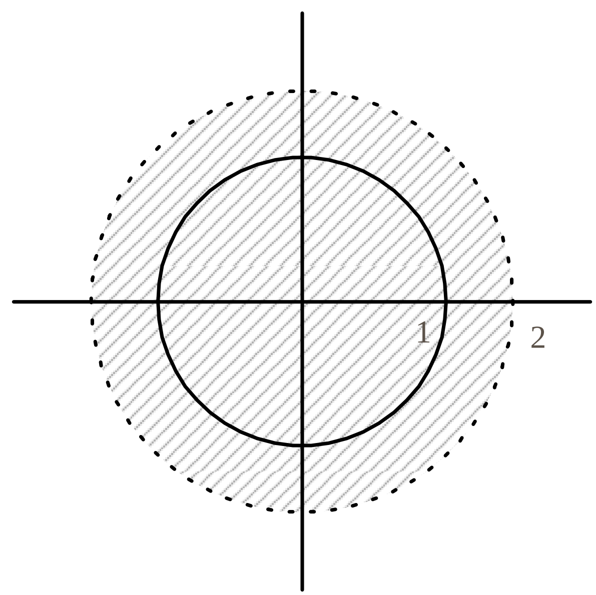
① **Anti Causal**

② **Stable**



$$X(z) = \sum_{n=-\infty}^{-1} -2^n z^{-n} = \frac{1}{1-2z^{-1}}$$

RoC : $|z| < 2$



① Inward

② UC $\subset$ RoC

What do you find?